Fourier Series and Fourier Transform

① $f_0 = 200\,\text{MHz}$, 50% duty cylce

$T_0 = \dfrac{1}{200\times10^6} = 5\times10^{-9} = 5\ \text{ns}.$ $\qquad \omega_0 = 2\pi f_0$

a) $C_n = \dfrac{1}{T_0}\displaystyle\int_{T_0} v(t)\,e^{-j\omega_0 n t}\,dt = \dfrac{1}{T_0}\int_0^{T_0/2} 5\,e^{-j2\pi\cdot 200\times10^6 n t}\,dt$

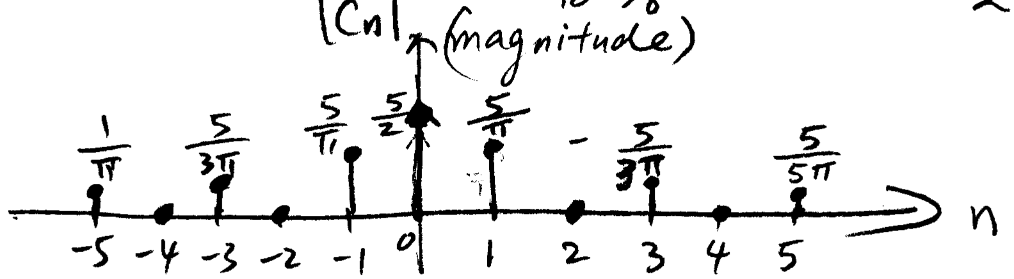
$= \dfrac{1}{T_0}\dfrac{5}{-j\omega_0 n}\displaystyle\int_0^{T_0/2} d\left(e^{-j\omega_0 n t}\right) = \dfrac{1}{T_0}\dfrac{5}{-j\omega_0 n}\,e^{-j\omega_0 n t}\Big|_0^{T_0/2}$

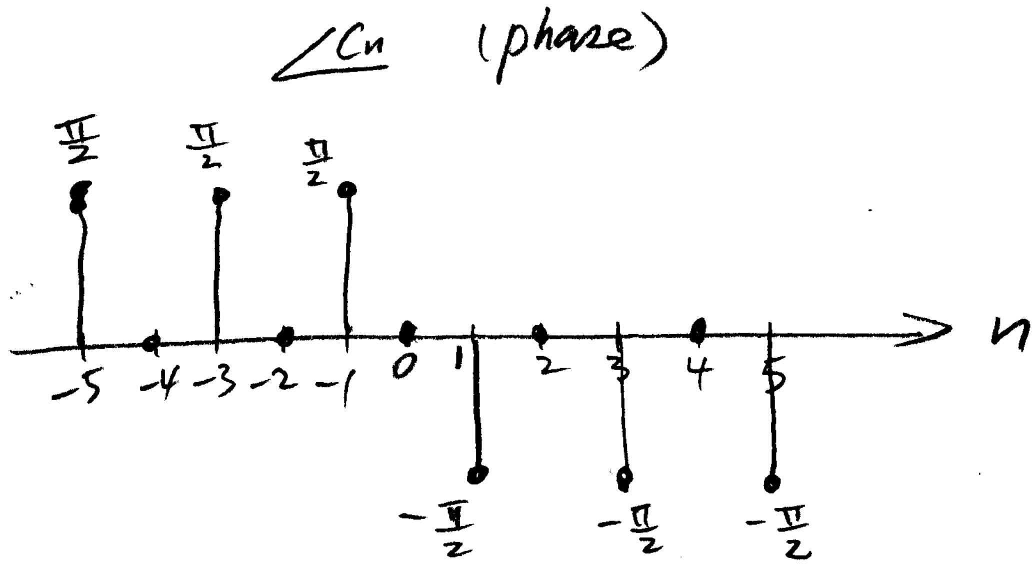
$= \dfrac{5}{-j2\pi n}\left(e^{-j\omega_0 n T_0/2} - 1\right) = \dfrac{5}{-j2\pi n}\left(e^{-j\pi n} - 1\right)$

$= j\dfrac{5}{2\pi n}\left((-1)^n - 1\right) \qquad \because e^{-j\pi} = -1$

$= \begin{cases} 0 & n = \pm2, \pm4, \cdots \text{(even)} \\ -j\dfrac{5}{\pi n} & n = \pm1, \pm3, \pm5, \cdots \text{(odd)} \end{cases}$

For $n = 0$, $\quad C_0 = \dfrac{1}{T_0}\displaystyle\int_0^{T_0/2} 5\,dt = \dfrac{5}{2}$





(b) $P_t = \frac{1}{T_0}\int_{T_0} v^2(t)\,dt = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} (25)\,dt = \frac{1}{T_0}(25)\cdot\frac{T_0}{2} = \frac{25}{2}$ ②

Power-partial $= |C_0|^2 + \sum_{n=1}^{5} 2|C_n|^2$

$= \left(\frac{5}{2}\right)^2 + 2\left(\frac{5}{\pi}\right)^2 + 2\left(\frac{5}{3\pi}\right)^2 + 2\left(\frac{1}{\pi}\right)^2 = 12.0616$

F.S.: $12.0616/12.5 = 96.65\%$ of power retained with $C_0, C_1, C_3, C_5, C_{-1}, C_{-3}, C_{-5}$.

(c) $x(t) = C_0 + 2|C_1|\cos(\omega_0 t + \angle C_1)$
$+ 2|C_3|\cos(3\omega_0 t + \angle C_3)$
$+ 2|C_5|\cos(5\omega_0 t + \angle C_5)$

$= \frac{5}{2} + \frac{10}{\pi}\cos(\omega_0 t - \frac{\pi}{2}) + \frac{10}{3\pi}\cos(3\omega_0 t - \frac{\pi}{2})$
$+ \frac{10}{5\pi}\cos(5\omega_0 t - \frac{\pi}{2})$ $\qquad \omega_0 = 2\pi\cdot 200\times10^6$ rad/s.

$= \frac{5}{2} + \frac{10}{\pi}\sin(\omega_0 t) + \frac{10}{3\pi}\sin(3\omega_0 t) + \frac{10}{5\pi}\sin(5\omega_0 t)$

The mean square error of this representation is
$12.5 - 12.0616 = 0.4384$ $(\text{unit})^2$ or 3.35% error

(2) $f(t) = 2 + 4\cos(20\pi t - 2) + \sin(60\pi t) + \frac{1}{4}\cos(80\pi t)$

Max Comm Factor of $\{20\pi, 60\pi, 80\pi\} = 20\pi$
$\omega_0 = 20\pi$ fundamental frequency

$f(t) = 2 + 4\cos(\omega_0 t - 2) + \cos(3\omega_0 t - \frac{\pi}{2}) + \frac{1}{4}\cos(4\omega_0 t)$

$C_0 = 2$; $|C_1| = \frac{4}{2} = 2$, $\angle C_1 = 2$; $|C_3| = \frac{1}{2}$, $\angle C_3 = -\frac{\pi}{2}$
$|C_4| = \frac{1}{8}$, $\angle C_4 = 0$, $\quad C_1 = 2e^{j2}$, $C_{-1} = 2e^{-j2}$
$C_3 = \frac{1}{2}e^{-j\frac{\pi}{2}} = -\frac{j}{2}$, $C_{-3} = \frac{j}{2}$, $C_4 = \frac{1}{4} = C_{-4}$

(3) $T_0 = \frac{2\pi}{\omega_0}$ $\int_{T_0} x(t) \cdot f^*(t)\, dt \overset{?}{=} 0$

check $\int_{T_0} e^{jk\omega_0 t} e^{-jn\omega_0 t} dt = \int_{T_0} e^{j(k-n)\omega_0 t} dt$

$= \frac{1}{j(k-n)\omega_0} e^{j(k-n)\omega_0 t} \Big|_0^{T_0} = \frac{1}{j(k-n)\omega_0} \Big(\underbrace{e^{j(k-n)2\pi}}_{=1} - \underbrace{e^{j0}}_{=1} \Big)$

$= 0$ Signal orthogonal when $k-n =$ integer, $k-n \neq 0$

when $k=n$

$\int_{T_0} e^{j(k-n)\omega_0 t} dt = \int_{T_0} dt = T_0 \neq 0$ the Signal is not orthogonal

4) $e^{-at}u(t) \Longleftrightarrow \frac{1}{a+j\omega}$

(a) $f(t) = 3e^{-3t}u(t-2) = 3e^{-3(t-2)}e^{-6}u(t-2)$

$= 3e^{-6} e^{-3(t-2)} u(t-2) \Longleftrightarrow F(j\omega) = \underline{\underline{\frac{3e^{-6}e^{-j2\omega}}{3+j\omega}}}$

(b) $x(t) = f(t) * \cos(4t)$

$X(j\omega) = F(\omega) \cdot \mathcal{F}\{\cos(4t)\} = \frac{2\pi}{a+j\omega} \cdot \left[\delta(\omega-4) + \delta(\omega+4)\right]$

$= \frac{2\pi}{a+j4}\delta(\omega-4) + \frac{2\pi}{a-j4}\delta(\omega+4)$

(c) $Y(\omega) = \text{rect}\left(\frac{3\omega - 9\pi}{6}\right)$. Using Duality

We have: $\tau\, \text{Sinc}\left(\frac{t\tau}{2}\right) \Longleftrightarrow 2\pi\, \text{rect}\left(\frac{\omega}{\tau}\right)$

④

Frequency shift property

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0)$$

$$Y(\omega) = \text{rect}\left(\frac{3(\omega-3\pi)}{6}\right) = \text{rect}\left(\frac{\omega-3\pi}{2}\right)$$

$\tau = 2 \quad \omega_0 = 3\pi$

$$y(t) = 2\,\text{sinc}(t)\,e^{j3\pi t} = (-1)^t \cdot 2\,\text{sinc}(t)$$

(5)

$$x(t) = c_0 + 2|c_2|\cos(2\cdot\omega_0 t + \angle c_2) + 2|c_5|\cos(5\omega_0 t + \angle c_5)$$

$|c_2| = 2, \quad \angle c_2 = -\frac{\pi}{2}$

$|c_5| = \sqrt{1^2+(\sqrt{3})^2} = 2, \quad \angle c_5 = \arctan(\sqrt{3}) = \frac{\pi}{3}$

$\angle c_5 = -\frac{\pi}{3}$

$$x(t) = 1 + 4\cos\left(20\pi t - \frac{\pi}{2}\right) + 4\cos\left(100\pi t - \frac{\pi}{3}\right)$$