Supplemental notes I

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@Mar. 2000 to June 2007

Sampling and discrete–time signals:

For every signal, there exists a Nyquist frequency $f_N = 2f_B$ where f_B (bandwidth) is the highest frequency component in a signal $x_a(t)$ which has significant energy. The sampling frequency $f_s > f_N$ to avoid aliasing error. $x(n) = x_a(t)|_{t=nT}$, where $T = \frac{1}{f_s}$ is the sampling interval. Typical discrete–time signals $u(n)$, $\delta(n)$,

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} = \sum_{k=-\infty}^{n} \delta(k), \qquad \delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{otherwise} \end{cases} = u(n) - u(n-1)$$

and

$$e^{j\omega_0 Tn} = e^{j\hat{\omega}_0 n} = e^{j\frac{2\pi l}{N}n} = \cos(\frac{2\pi l}{N}n) + j\sin(\frac{2\pi l}{N}n)$$

where N is the period of the complex exponential signal and $\hat{\omega}_0 = \omega_0 T = \frac{2\pi l}{N}$ is the normalized frequency in radian. $e^{j\hat{\omega}_0 n}$ is periodic, only if $\frac{l}{N}$ is rational, that is, l and N are both integers.

Fourier series:

For any periodic signal $x_a(t)$ with period T_0, the Fourier series representation of the signal is

$$x_a(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 kt} \tag{1}$$

and

$$X_k = \frac{1}{T_0}\int_{T_0} x_a(t) e^{-j\omega_0 kt} dt \tag{2}$$

where $\omega_0 = \frac{2\pi}{T_0}$. A periodic signal can be expressed as a linear combination of sinusoids with frequencies of $k\omega_0$. Therefore, the ratio of any two frequency components in a periodic signal must be a rational number,

$$\frac{\omega_1}{\omega_2} = \frac{l\omega_0}{m\omega_0} = \frac{l}{m}$$

and its period is $T_0 = \frac{2\pi}{\omega_0}$ where ω_0 is a common factor of the frequency components $\omega_1, \omega_2, \omega_3, \cdots$.

Example 1 *An analog signal $x_a(t) = 3\cos(300\pi t + 2) + 2\cos(450\pi t)$ is periodic, since*

$$\frac{\omega_1}{\omega_2} = \frac{300\pi}{450\pi} = \frac{2\times 150\pi}{3\times 150\pi} \ or \ \frac{6\times 50\pi}{9\times 50\pi}\ ,$$

and the period is either $T_0 = \frac{2\pi}{\omega_0} = \frac{1}{75}$ or $\frac{1}{25}$ second and $\omega_0 = 150\pi$ or 50π respectively.

Example 2 *The bandwidth of $x_a(t) = 3\cos(300\pi t + 2) + 2\cos(450\pi t)$ is $f_B = 225$ Hz. Therefore, the Nyquist frequency of the signal is $f_N = 2f_B = 450$ Hz. To satisfy sampling theory, we select a sampling frequency of $f_s = 500$ Hz that is greater than f_N.*

$$\begin{aligned} x(n) &= x_a(t)|_{t=nT} = 3\cos(\frac{300\pi n}{500} + 2) + 2\cos(\frac{450\pi n}{500}) \\ &= 3\cos(\frac{6*2\pi n}{20} + 2) + 2\cos(\frac{9*2\pi n}{20}) \end{aligned}$$

where the normalized frequencies $\hat{\omega}_1 = \omega_1 T = \frac{l_1 2\pi}{N}$ and $\hat{\omega}_2 = \omega_2 T = \frac{l_2 2\pi}{N}$. The signal is periodic and the period is $N = 20$ samples. One period (20 samples) of the digital signal covers 6 (l_1) period of frequency component of 300π (rad./sec.) and 9 (l_2) period of frequency component of 450π (rad./sec.). The 20 sample period of the discrete–time signal is equivalent to $20T = \frac{1}{25}$ second.

Discrete Fourier series (DFS) of a periodic signal:

A periodic signal can be expressed by a linear combination of sinusoids (1). Sampling of sinusoids in the linear combination generates a set of normalized frequency components, such as, $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{\omega}_3$, $\cdots$. The frequency components can be expressed as $\hat{\omega}_1 = \frac{l_1 2\pi}{N}$, $\hat{\omega}_2 = \frac{l_2 2\pi}{N}$, $\hat{\omega}_3 = \frac{l_3 2\pi}{N}$, $\cdots$, where N is the period. The fundamental frequency is $\hat{\omega}_0 = \frac{2\pi}{N}$. If a period of the sampled discrete–time signal $x(n)$ is used for Fourier series analysis, the equations (1) and (2) yield

$$\begin{aligned} x(n) &= x_a(nT) = \sum_{k=0}^{N-1} X_k e^{j\omega_0 knT} \\ &= \sum_{k=0}^{N-1} X_k e^{j\hat{\omega}_0 kn} = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1, \cdots, N-1 \end{aligned} \tag{3}$$

and

$$\begin{aligned} X_k &= \frac{1}{T_0} \int_{T_0} x_a(nT) e^{-j\omega_0 knT} T \\ &= \frac{1}{NT} \sum_{n=0}^{N-1} x_a(nT) e^{-j\omega_0 knT} T \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \cdots, N-1 \end{aligned} \tag{4}$$

respectively. The convention used in Matlab^{TM} and other text book is as follows.

$$\text{DFS: } X(k) = X_k \cdot N = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \cdots, N-1 \tag{5}$$

and

$$\text{IDFS: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1, \cdots, N-1. \tag{6}$$

There are three important properties of DFS:

Priodicity: $X(k) = X(k \pm N)$ and $x(n) = x(n \pm N)$

Symmetry: $X(k) = X^*(N-k)$, if $x(n)$ is a real valued signal.

Example 3 *Consider a periodic signal*

$$\begin{aligned} x_a(t) &= 10 + 14\cos(200\pi t - \frac{\pi}{3}) + 8\sin(500\pi t + \pi) \\ &= 10 + 14\frac{e^{j(200\pi t - \frac{\pi}{3})} + e^{-j(200\pi t - \frac{\pi}{3})}}{2} + 8\frac{e^{j(500\pi t + \pi)} - e^{-j(500\pi t + \pi)}}{2j} \\ &= 10 + 7e^{-j\frac{\pi}{3}} \cdot e^{j200\pi t} + 7e^{j\frac{\pi}{3}} \cdot e^{-j200\pi t} + 4e^{-j\frac{\pi}{2}}(e^{j\pi} \cdot e^{j500\pi t} + e^{-j500\pi t}) \\ &= 10 + 7e^{-j\frac{\pi}{3}} \cdot e^{j200\pi t} + 7e^{j\frac{\pi}{3}} \cdot e^{-j200\pi t} + 4e^{j\frac{\pi}{2}} \cdot e^{j500\pi} + 4e^{-j\frac{\pi}{2}} \cdot e^{-j500\pi}. \end{aligned}$$

We can find the Fourier series coefficients X_k and ω_0 by using series recognition method.

$$\frac{\omega_1}{\omega_2} = \frac{200\pi}{500\pi} = \frac{2 \times 100\pi}{5 \times 100\pi}, \quad \omega_0 = 100\pi$$

and

$$X_0 = 10; \quad X_2 = X_{-2}^* = 7e^{-j\frac{\pi}{3}}; \quad X_5 = X_{-5}^* = 4e^{j\frac{\pi}{2}}$$

Since the Nyquist frequency of the signal is $f_N = 500$ Hz, the $f_s = 600$ Hz is selected. Therefore, the normalized frequency components are

$$\hat{\omega}_0 = \frac{100\pi}{600} = \frac{2\pi}{12}; \quad \hat{\omega}_1 = \frac{200\pi}{600} = \frac{2 \cdot 2\pi}{12}; \quad \hat{\omega}_2 = \frac{500\pi}{600} = \frac{5 \cdot 2\pi}{12}$$

and the period of the sampled signal is $N = 12$.

$$\begin{aligned} x(n) &= 10 + 14\cos(\frac{2\cdot 2\pi}{12}n - \frac{\pi}{3}) + 8\sin(\frac{5\cdot 2\pi}{12}n + \pi) \\ &= 10 + 7e^{-j\frac{\pi}{3}}\cdot e^{j\frac{2\cdot 2\pi}{12}n} + 7e^{j\frac{\pi}{3}}\cdot e^{-j\frac{2\cdot 2\pi}{12}n} + 4e^{j\frac{\pi}{2}}\cdot e^{j\frac{5\cdot 2\pi}{12}n} + 4e^{-j\frac{\pi}{2}}\cdot e^{-j\frac{5\cdot 2\pi}{12}n} \end{aligned}$$

where $n = 0, 1, \cdots, 11$ *covers one period of* $x(n)$. *In order to find the discrete Fourier series of the above signal, we need to use the following DFS pair:*

$$e^{j\frac{l\cdot 2\pi}{N}n}, \quad n = 0, 1, \cdots, N-1 \Longleftrightarrow N\cdot\delta(k-l), \quad k = 0, 1, \cdots, N-1 \tag{7}$$

and it also implies

$$1, \quad n = 0, 1, \cdots, N-1 \Longleftrightarrow N\cdot\delta(k), \quad k = 0, 1, \cdots, N-1$$

the proof will be in the next example. Applying equation (7) to $x(n)$, *it yields*

$$\begin{aligned} X(k) &= [10\cdot\delta(k) + 7e^{-j\frac{\pi}{3}}\cdot\delta(k-2) + 7e^{j\frac{\pi}{3}}\cdot\delta(k+2) + 4e^{j\frac{\pi}{2}}\cdot\delta(k-5) + 4e^{-j\frac{\pi}{2}}\cdot\delta(k+5)]\cdot N \\ &= 120\cdot\delta(k) + 84e^{-j\frac{\pi}{3}}\cdot\delta(k-2) + 84e^{j\frac{\pi}{3}}\cdot\delta(k-N+2) + 48e^{j\frac{\pi}{2}}\cdot\delta(k-5) + 48e^{-j\frac{\pi}{2}}\cdot\delta(k-N+5) \\ &= 120\cdot\delta(k) + 84e^{-j\frac{\pi}{3}}\cdot\delta(k-2) + 84e^{j\frac{\pi}{3}}\cdot\delta(k-10) + 48e^{j\frac{\pi}{2}}\cdot\delta(k-5) + 48e^{-j\frac{\pi}{2}}\cdot\delta(k-7) \end{aligned}$$

where the periodicity property is used and $N = 12$ *is the period. The result can be verified by the attached Matlab*^{TM} *code.*

k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*(-5)	*8*(-4)	*9*(-3)	*10*(-2)	*11*(-1)
$\omega = k\cdot\omega_0$	*0*	100π	200π	300π	400π	500π	600π	-500π	-400π	-300π	-200π	-100π
$\hat{\omega} = k\cdot\frac{2\pi}{N}$	*0*	$\frac{2\pi}{12}$	$\frac{4\pi}{12}$	$\frac{6\pi}{12}$	$\frac{8\pi}{12}$	$\frac{10\pi}{12}$	π	$-\frac{10\pi}{12}$	$-\frac{8\pi}{12}$	$-\frac{6\pi}{12}$	$-\frac{4\pi}{12}$	$-\frac{2\pi}{12}$
$X(k)$	*120*	*0*	$84e^{-j\frac{\pi}{3}}$	*0*	*0*	$48e^{j\frac{\pi}{2}}$	*0*	$48e^{-j\frac{\pi}{2}}$	*0*	*0*	$84e^{j\frac{\pi}{3}}$	*0*
X_k	*10*	*0*	$7e^{-j\frac{\pi}{3}}$	*0*	*0*	$4e^{j\frac{\pi}{2}}$	*0*	$4e^{-j\frac{\pi}{2}}$	*0*	*0*	$7e^{j\frac{\pi}{3}}$	*0*
$X(k)$ *fft(·)*	*120*	*0*	$42 - j72.8$	*0*	*0*	$48j$	*0*	$-48j$	*0*	*0*	$42 + j72.8$	*0*
$X_k = \frac{X(k)}{N}$	*10*	*0*	$3.5 - j6.1$	*0*	*0*	$4j$	*0*	$-4j$	*0*	*0*	$3.5 + j6.1$	*0*

The hand calculated and computer generated results are compare in the table.

Matlab^{TM} code list:

```
% Discrete Fourier Series Example
% Kefu Xue, Ph.D.
% @ March 2000
%
fs=600;
T=1/fs;
n=0:11;
t=n*T;
xn=10+14*cos(200*pi*t-pi/3)+8*sin(500*pi*t+pi);
Xok=fft(xn);
Xk=Xok/12;
```

The complex value of hand calculated coefficient, $84e^{-j\frac{\pi}{3}}$, can be verified in Matlab^{TM} as follows.

```
84*exp(-j*pi/3)
```

Example 4 *Find DFS of* $e^{j\frac{l\cdot 2\pi}{N}n}$, $\sin(\frac{l\cdot 2\pi}{N}n)$ *and* $\cos(\frac{l\cdot 2\pi}{N}n)$, $\quad n = 0, 1, \cdots, N-1$.

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} e^{j\frac{l\cdot 2\pi}{N}n} e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \cdots, N-1 \\ &= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)n} = \begin{cases} \sum_{n=0}^{N-1} 1, & \text{for } k = l \\ \frac{1-e^{-j\frac{2\pi}{N}(k-l)N}}{1-e^{-j\frac{2\pi}{N}(k-l)}}, & \text{for } k \neq l \end{cases} \\ &= \begin{cases} N, & \text{for } k = l \\ 0, & \text{for } k \neq l \end{cases} = N\cdot\delta(k-l), \quad k = 0, 1, \cdots, N-1 \end{aligned}$$

In the case of $\cos(\frac{l\cdot 2\pi}{N}n)$*, we have*

$$\cos(\frac{l\cdot 2\pi}{N}n) = \frac{1}{2}[e^{j\frac{l\cdot 2\pi}{N}n} + e^{-j\frac{l\cdot 2\pi}{N}n}].$$

Using the complex exponential DFS pair, we have

$$X(k) = \frac{N}{2}[\delta(k-l) + \delta(k+l)]$$

Since $X(k) = X(k-N)$ *is periodic, it yields the following DFS pair.*

$$\cos(\frac{l\cdot 2\pi}{N}n), \quad n = 0, 1, \cdots, N-1 \Longleftrightarrow \frac{N}{2}[\delta(k-l) + \delta(k-N+l)], \quad k = 0, 1, \cdots, N-1 \tag{8}$$

Similarly,

$$\sin(\frac{l\cdot 2\pi}{N}n) = \frac{1}{2j}[e^{j\frac{l\cdot 2\pi}{N}n} - e^{-j\frac{l\cdot 2\pi}{N}n}]$$

yields

$$\sin(\frac{l\cdot 2\pi}{N}n), \quad n = 0, 1, \cdots, N-1 \Longleftrightarrow \frac{N}{2j}[\delta(k-l) - \delta(k-N+l)], \quad k = 0, 1, \cdots, N-1 \tag{9}$$

Time–Frequency Spectrum

Consider a piece of music: "C(middle C), C, D, E, C, E, D, $\underset{\cdot}{G}$ (on octave down)", where middle C=262 Hz, D=294 Hz, E=330 Hz and $\underset{\cdot}{G}$ (one octave down from middle G)=196 Hz. The fundamental frequency for this piece of music is 2 Hz which is the maximum common factor of 262, 294, 330, 196. If a sampling frequency $f_s = 1000$ Hz is selected, $\frac{f_s}{f_0} = 500$ will cover one period of the music, that is, $N = 500$. The music is formed as follows.

$$\begin{aligned}
x(n) &= \cos(\frac{2\pi C}{1000}n) \quad (n = 0, \cdots, 499) \\
+0's \quad (n &= 500, \cdots, 999) + \cos(\frac{2\pi C}{1000}n) \quad (n = 1000, \cdots, 1499) \\
+0's \quad (n &= 1500, \cdots, 1999) + \cos(\frac{2\pi C}{1000}n) \quad (n = 2000, \cdots, 2499) \\
+0's \quad (n &= 2500, \cdots, 2999) + \cos(\frac{2\pi D}{1000}n) \quad (n = 3000, \cdots, 3499) \\
&\cdots \\
+0's \quad (n &= 6500, \cdots, 6999) + \cos(\frac{2\pi \underset{\cdot}{G}}{1000}n) \quad (n = 7000, \cdots, 7499)
\end{aligned}$$

In the music $x(n)$, each note plays for 0.5 second and is separated with a silent space for 0.5 second. The Fourier spectrum shows all 4 frequency components in figure 1 (top–left). In order to reflect when a particular note is played, the spectrogram (DFS) is calculated with a non–overlap sliding window of length 500 samples (0.5 second). For each 500 samples of signal, we have the spectrum calculated and the spectra are displayed in figure 1. The Matlab^{TM} program of this example can be found on the Web site.

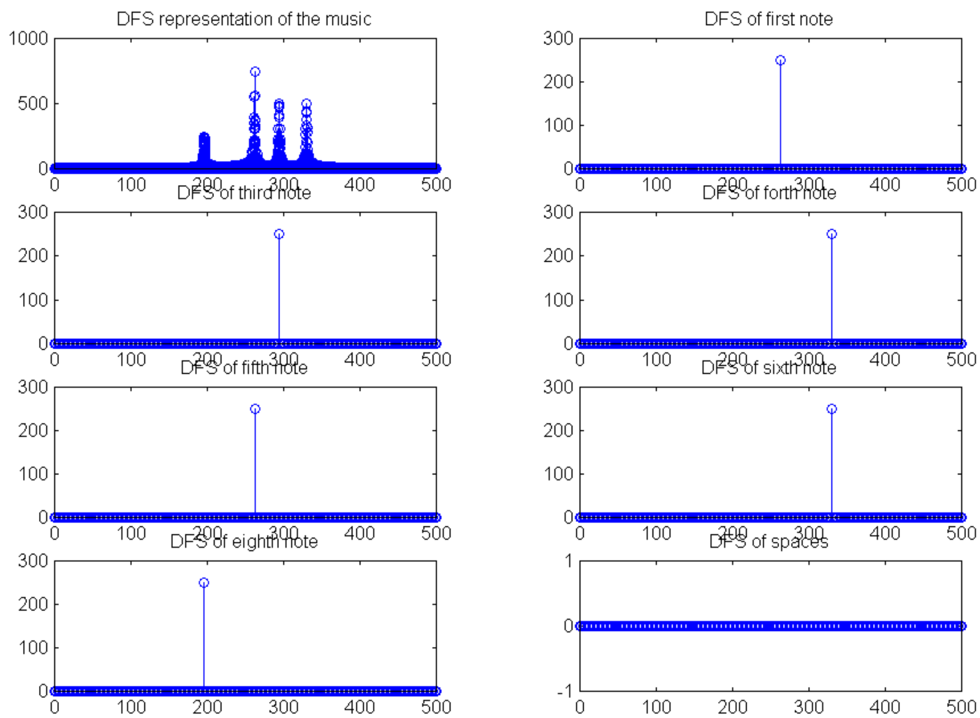

Figure 1: The comparison of spectrum and spectrogram of the music