

Chapter 5 and 6 supplemental notes

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Discrete-time linear and time-invariant (LTI) systems: Difference equation, Transfer function, impulse response, and Frequency response.

- Difference equation:

$$y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_Lx(n-L)$$

- Short form expression of difference equation:

$$\{y(n)\} = \sum_{k=0}^L b_k z^{-k} \{x(n)\}, \text{ and, } \{y(n)\} = B(z)\{x(n)\}$$

- The transfer function:

$$H(z) \triangleq \frac{Y(z)}{X(z)} = B(z) = \sum_{n=0}^L b_n z^{-n}, \quad h(n) = b_n, \text{ for } n = 0, 1, \dots, L$$

- The frequency response

$$H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}} = B(e^{j\hat{\omega}}) = \sum_{n=0}^L h(n)e^{-j\hat{\omega}n}$$

- Linear convolution sum: For a LTI system, the forced (zero-state) response can be expressed as

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k).$$

For a finite length linear convolution sum, the length L , the starting S and ending E indices can be decided by

$$L_y = L_h + L_x - 1; \quad S_y = S_h + S_x; \quad E_y = E_h + E_x; \quad L_y = E_y - S_y + 1.$$

Linearity: $x_1(n) \rightarrow y_1(n)$; $x_2(n) \rightarrow y_2(n)$; the system is linear, iff $x_3(n) = \alpha_1x_1(n) + \alpha_2x_2(n) \rightarrow y_3(n) = \alpha_1y_1(n) + \alpha_2y_2(n)$.

Time-invariant: $x_1(n) \rightarrow y_1(n)$; the system is time-invariant, iff $x_2(n) = x_1(n-n_0) \rightarrow y_2(n) = y_1(n-n_0)$.

Causality: The output $y(n)$ does not precede the input $x(n)$. For a causal LTI system, $h(n) = 0$ for $n < 0$.

Sinusoidal steady-state response of LTI systems: Given eigenfunction (complex exponential excitation) $e^{j\hat{\omega}_0 n}$,

$$y_{ss}(n) = H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n} = \left| H(e^{j\hat{\omega}_0}) \right| e^{j\angle H(e^{j\hat{\omega}_0})} e^{j\hat{\omega}_0 n} = \left| H(e^{j\hat{\omega}_0}) \right| e^{j[\hat{\omega}_0 n + \angle H(e^{j\hat{\omega}_0})]}$$

where $H(e^{j\hat{\omega}_0}) = \left| H(e^{j\hat{\omega}_0}) \right| e^{j\angle H(e^{j\hat{\omega}_0})}$ is referred to as the eigenvalue. For real sinusoidal excitations $\cos(\hat{\omega}_0 n + \varphi)$ and $\sin(\hat{\omega}_0 n + \varphi)$,

$$y_{ss}(n) = \left| H(e^{j\hat{\omega}_0}) \right| \cos(\hat{\omega}_0 n + \varphi + \angle H(e^{j\hat{\omega}_0}))$$

and

$$y_{ss}(n) = \left| H(e^{j\hat{\omega}_0}) \right| \sin(\hat{\omega}_0 n + \varphi + \angle H(e^{j\hat{\omega}_0})) = \left| H(e^{j\hat{\omega}_0}) \right| \cos(\hat{\omega}_0 n + \varphi - \frac{\pi}{2} + \angle H(e^{j\hat{\omega}_0}))$$

respectively. When $\angle H(e^{j\hat{\omega}}) \equiv 0$, the frequency response $H(e^{j\hat{\omega}})$ is a real valued function and referred to as zero-phase filter. When $\angle H(e^{j\hat{\omega}}) = -\alpha\hat{\omega} + \beta$, the frequency response $H(e^{j\hat{\omega}})$ is referred to as linear-phase filter where α and β are real valued constants. $\left| H(e^{j\hat{\omega}}) \right|$ describes the frequency selectivities of a digital filter. The group delay of a digital filter is defined as $\tau(\hat{\omega}) = -\frac{d\angle H(e^{j\hat{\omega}})}{d\hat{\omega}}$ (samples) that indicates the time delays ($\tau(\hat{\omega}) \cdot T$) for various frequency components in the filtered signal. For a linear phase filter, the group delay $\tau(\hat{\omega}) = \alpha$ is a constant for all the frequency components of the filtered signal.

Sketch the magnitude frequency response from the pole/zero diagram:

$$H(z) = B(z) = b_0 z^{-L} \prod_{k=1}^L (z - z_k)$$

where z_k and p_k are the zeros and poles of the system respectively.

$$\left| H(e^{j\hat{\omega}}) \right| = |b_0| \prod_{k=1}^L \left| e^{j\hat{\omega}} - z_k \right| = |b_0| \prod_{k=1}^L B_k$$

where $B_k = \left| e^{j\hat{\omega}} - z_k \right|$ is the distances from a frequency $0 \leq \hat{\omega} \leq \pi$ to zero z_k . Note that $|H(1)|$ is the DC gain ($\hat{\omega} = 0$), $|H(j)|$ is the gain at $\hat{\omega} = \frac{\pi}{2}$ and $|H(-1)|$ is the gain at $\hat{\omega} = \pi$.

Linear phase filter properties:

- Impulse response sequence is even or odd symmetry: $h(n) = h(N-1-n)$ or $h(n) = -h(N-1-n)$, where N is the filter length.
- For N is odd and $h(n) = h(N-1-n)$, it is a type I linear phase filter,

$$H(z) = h\left(\frac{N-1}{2}\right) z^{-\frac{N-1}{2}} + \sum_{n=0}^{\frac{N-1}{2}-1} h(n) [z^{-n} + z^{-(N-1-n)}],$$

$$H(e^{j\hat{\omega}}) = \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-1}{2}-1} 2h(n) \cos\left(\frac{N-1}{2} - n\right)\hat{\omega} \right] \exp(-j\frac{N-1}{2}\hat{\omega}).$$

- For N is even and $h(n) = h(N-1-n)$, it is a type II linear phase filter,

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n) [z^{-n} + z^{-(N-1-n)}], \quad H(e^{j\hat{\omega}}) = \sum_{n=0}^{\frac{N}{2}-1} 2h(n) [\cos\left(\frac{N-1}{2} - n\right)\hat{\omega}] \exp(-j\frac{N-1}{2}\hat{\omega}),$$

where $\left| H(e^{j\hat{\omega}}) \right|_{\hat{\omega}=\pi} \equiv 0$ independent of filter coefficients $h(n)$.

- The phase function and group delay function of a linear phase filter are independent of filter coefficients $h(n)$,

$$\angle H(e^{j\hat{\omega}}) = -\frac{N-1}{2}\hat{\omega} + \beta; \quad \tau(\hat{\omega}) = \frac{N-1}{2} \text{ where } \beta \text{ is a real constant value.}$$

Linear phase comb (Notch) filters:

- $h(n) = \delta(n) - \delta(n-L)$, (length $N = L+1$); $H(z) = 1 - z^{-L}$; and

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}L} = e^{-j\frac{\hat{\omega}L}{2}}(e^{j\frac{\hat{\omega}L}{2}} - e^{-j\frac{\hat{\omega}L}{2}}) = 2\sin\left(\frac{\hat{\omega}L}{2}\right)e^{-j\frac{\hat{\omega}L}{2} + \frac{\pi}{2}}.$$

It is a linear phase filter with group delay $\tau(\hat{\omega}) = \frac{L}{2} = \frac{N-1}{2}$ and zeros (notches at) $\frac{\hat{\omega}L}{2} = 0$ or $l \cdot \pi$, $l = 0, 1, \dots, L$, that is

$$\hat{\omega} = 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \dots, \frac{l \cdot 2\pi}{L}, \dots, \pi. \quad \text{in real frequencies: } f = \frac{l \cdot f_s}{L}, \quad l = 0, 1, \dots, \left\lfloor \frac{L}{2} \right\rfloor.$$

- $h(n) = \delta(n) + \delta(n-L)$, (length $N = L+1$); $H(z) = 1 + z^{-L}$; and

$$H(e^{j\hat{\omega}}) = 1 + e^{-j\hat{\omega}L} = e^{-j\frac{\hat{\omega}L}{2}}(e^{j\frac{\hat{\omega}L}{2}} + e^{-j\frac{\hat{\omega}L}{2}}) = 2\cos\left(\frac{\hat{\omega}L}{2}\right)e^{-j\frac{\hat{\omega}L}{2}}.$$

It is a linear phase filter with group delay $\tau(\hat{\omega}) = \frac{N-1}{2}$ and zeros (notches at) $\frac{\hat{\omega}L}{2} = \frac{l \cdot \pi}{2}$, $l = 1, 3, 5, \dots$, odd integer $< 2L$, that is

$$\hat{\omega} = \frac{\pi}{L}, \frac{3\pi}{L}, \dots, \frac{l \cdot \pi}{L}, \dots, \pi. \quad \text{in real frequencies: } f = \frac{l \cdot f_s}{2L}, \quad l = 1, 3, 5, \dots, \text{odd integer } \leq L.$$

Supplemental problems and computer project:

1. A LTI system is described by the following impulse response

$$h(n) = 2^{-n} \quad \text{for } n = -2, -1, \dots, 3, 4$$

For a given input signal

$$\left\{ \begin{array}{|c|c|c|c|} \hline x(n) = & 1 & -1 & 1 & 2 \\ \hline n = & 3 & 4 & 5 & 6 \\ \hline \end{array} \right\},$$

answer the following questions.

- (a) Find the difference equation representation of the filter and system transfer function $B(z)$.
- (b) What is the length of the zero-state response sequence $y(n)$?
- (c) What are the beginning and ending indices of the zero-state response $y(n)$?
- (d) Find some of the values of the zero-state response sequence

$$y(-2), y(0), y(4), y(8), y(11)$$

using the linear convolution sum.

- (e) Verify your result using MatlabTM function conv().

2. A linear time-invariant continuous-time integrator is described by the following differential equation.

$$y_a(t) = \frac{d^2 x_a(t)}{dt^2} - 5 \frac{dx_a(t)}{dt} + x_a(t)$$

Find the corresponding difference equation with input $x(n)$, output $y(n)$ and $B(z)$ using the approximation,

$$\frac{dx_a(t)}{dt} \approx \frac{x_a(nT) - x_a(nT - T)}{T} = \frac{x(n) - x(n-1)}{T}.$$

Hint: $\frac{d^2 x_a(t)}{dt^2} = \frac{d}{dt} \left\{ \frac{dx_a(t)}{dt} \right\}$.

3. Given the zeros of a FIR digital filter as follows: $z_0 = -1$, $z_{1,2} = e^{\pm j \frac{\pi}{4}}$.
- Draw the pole/zero diagram.
 - Find the difference equation representation of the filter and system transfer function $B(z)$. Can you judge the filter is a linear phase filter or not from its filter coefficient $h(n)$? Is it?
 - Sketch the magnitude frequency response $|H(e^{j\hat{\omega}})|$ with the values clearly marked at $\hat{\omega} = 0, \frac{\pi}{2}$ and π .
 - Pick up a proper sampling frequency f_s such that the frequency components 200π and 800π in the following signal will be filtered out by the digital filter.

$$x_a(t) = 0.2 \sin(200\pi t + 1) - 0.3 \cos(800\pi t - 1.5) + 2 \cos(70\pi t - 1)$$

and find the system steady state response $y_{ss}(n)$ to the signal.

- Find the expression of the phase frequency response $\angle H(e^{j\hat{\omega}})$ and group delay $\tau(\hat{\omega})$.
 - Verify the results using MatlabTM. You may use functions such as `filter()`, `freqz()`, `poly()`, `fft()`, `abs()`, `angle()`, `impz()` and `zplane()`, etc.
4. In the following signal

$$x_a(t) = e^{-0.5t} + 0.5 \cos(400\pi t - 1) + 0.03 \cos(1200\pi t - \frac{\pi}{2})$$

where $0 \leq t \leq 2$, the sinusoids are considered interferences to a low-pass type signal $e^{-0.5t}$. Design a Notch (Nulling) filter $B(z) = 1 + z^{-L}$ and a proper sampling frequency to eliminate the sinusoidal interferences.

- Show the pole/zero diagram using `zplane()` function.
- Plot frequency response (both magnitude and phase) of the filter in linear scale with frequency label in Hz. Plot the magnitude of the Fourier transform of $x(n)$ with frequency axis labeled in Hz, that is, $k=0:N-1$; $f=k*fs/N$; where N is the length of $x(n)$.
- Using `subplot()` to show the plots of the original signal $x_a(t)$, filtered signal $y(n)$, and desired signal $e^{-0.5t}$.
- Plot the magnitude of the Fourier transform of the filtered signal $y(n)$ with frequency axis labeled in Hz.
- Draw conclusions from the observation.