Chapter 5 and 6 supplemental notes

Instructor: Kefu Xue, Ph. D., Dept. of EE, WSU

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Discrete–time linear and time–invariant (LTI) systems: Difference equation, Transfer function, impulse response, and Frequency response.

- Difference equation:
$$y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_Lx(n-L)$$

- Short form expression of difference equation:
$$\{y(n)\} = \sum_{k=0}^{L} b_k z^{-k}\{x(n)\}, \text{ and, } \{y(n)\} = B(z)\{x(n)\}$$

- The transfer function:
$$H(z) \triangleq \frac{Y(z)}{X(z)} = B(z) = \sum_{n=0}^{L} b_n z^{-n},\ h(n) = b_n, \text{ for } n = 0, 1, \cdots, L$$

- The frequency response
$$H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}} = B(e^{j\hat{\omega}}) = \sum_{n=0}^{L} h(n)e^{-j\hat{\omega}n}$$

- Linear convolution sum: For a LTI system, the forced (zero–state) response can be expressed as
$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k).$$

  For a finite length linear convolution sum, the length L, the starting S and ending E indices can be decided by
$$L_y = L_h + L_x - 1; \quad S_y = S_h + S_x; \quad E_y = E_h + E_x; \quad L_y = E_y - S_y + 1.$$

Linearity: $x_1(n) \to y_1(n)$; $x_2(n) \to y_2(n)$; the system is linear, iff $x_3(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) \to y_3(n) = \alpha_1 y_1(n) + \alpha_2 y_2(n)$.
Time–invariant: $x_1(n) \to y_1(n)$; the system is time–invariant, iff $x_2(n) = x_1(n-n_0) \to y_2(n) = y_1(n-n_0)$.
Causality: The output $y(n)$ does not precede the input $x(n)$. For a causal LTI system, $h(n) = 0$ for $n < 0$.

Sinusoidal steady–state response of LTI systems: Given eigenfunction (complex exponential excitation) $e^{j\hat{\omega}_0 n}$,

$$y_{ss}(n) = H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n} = \left|H(e^{j\hat{\omega}_0})\right| e^{j\angle H(e^{j\hat{\omega}_0})}e^{j\hat{\omega}_0 n} = \left|H(e^{j\hat{\omega}_0})\right| e^{j[\hat{\omega}_0 n+\angle H(e^{j\hat{\omega}_0})]}$$

where $H(e^{j\hat{\omega}_0}) = \left|H(e^{j\hat{\omega}_0})\right| e^{j\angle H(e^{j\hat{\omega}_0})}$ is referred to as the eigenvalue. For real sinusoidal excitations $\cos(\hat{\omega}_0 n+\varphi)$ and $\sin(\hat{\omega}_0 n+\varphi)$,

$$y_{ss}(n) = \left|H(e^{j\hat{\omega}_0})\right| \cos(\hat{\omega}_0 n+\varphi+\angle H(e^{j\hat{\omega}_0}))$$

and

$$y_{ss}(n) = \left|H(e^{j\hat{\omega}_0})\right| \sin(\hat{\omega}_0 n+\varphi+\angle H(e^{j\hat{\omega}_0})) = \left|H(e^{j\hat{\omega}_0})\right| \cos(\hat{\omega}_0 n+\varphi-\frac{\pi}{2}+\angle H(e^{j\hat{\omega}_0}))$$

respectively. When $\angle H(e^{j\hat{\omega}}) \equiv 0$, the frequency response $H(e^{j\hat{\omega}})$ is a real valued function and referred to as zero–phase filter. When $\angle H(e^{j\hat{\omega}}) = -\alpha\hat{\omega}+\beta$, the frequency response $H(e^{j\hat{\omega}})$ is referred to as linear–phase filter where α and β are real valued constants. $\left|H(e^{j\hat{\omega}})\right|$describes the frequency selectivities of a digital filter. The group delay of a digital filter is defined as $\tau(\hat{\omega}) = -\frac{d\angle H(e^{j\hat{\omega}})}{d\hat{\omega}}$ (samples) that indicates the time delays $(\tau(\hat{\omega})\cdot T)$ for various frequency components in the filtered signal. For a linear phase filter, the group delay $\tau(\hat{\omega}) = \alpha$ is a constant for all the frequency components of the filtered signal.

Sketch the magnitude frequency response from the pole/zero diagram:

$$H(z) = B(z) = b_0 z^{-L}\prod_{k=1}^{L}(z-z_k)$$

where z_k and p_k are the zeros and poles of the system respectively.

$$\left|H(e^{j\hat{\omega}})\right| = |b_0|\prod_{k=1}^{L}\left|e^{j\hat{\omega}}-z_k\right| = |b_0|\prod_{k=1}^{L}B_k$$

where $B_k = \left|e^{j\hat{\omega}}-z_k\right|$ is the distances from a frequency $0\le\hat{\omega}\le\pi$ to zero z_k. Note that $|H(1)|$ is the DC gain $(\hat{\omega}=0)$, $|H(j)|$ is the gain at $\hat{\omega}=\frac{\pi}{2}$ and $|H(-1)|$ is the gain at $\hat{\omega}=\pi$.

Linear phase filter properties:

- Impulse response sequence is even or odd symmetry: $h(n) = h(N-1-n)$ or $h(n) = -h(N-1-n)$, where N is the filter length.
- For N is odd and $h(n) = h(N-1-n)$, it is a type I linear phase filter,

$$H(z) = h(\frac{N-1}{2})z^{-\frac{N-1}{2}} + \sum_{n=0}^{\frac{N-1}{2}} h(n)[z^{-n}+z^{-(N-1-n)}],$$

$$H(e^{j\hat{\omega}}) = [h(\frac{N-1}{2}) + \sum_{n=0}^{\frac{N-1}{2}} 2h(n)\cos(\frac{N-1}{2}-n)\hat{\omega}]\exp(-j\frac{N-1}{2}\hat{\omega}).$$

- For N is even and $h(n) = h(N-1-n)$, it is a type II linear phase filter,

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n)[z^{-n}+z^{-(N-1-n)}], \quad H(e^{j\hat{\omega}}) = \sum_{n=0}^{\frac{N}{2}-1} 2h(n)[\cos(\frac{N-1}{2}-n)\hat{\omega}]\exp(-j\frac{N-1}{2}\hat{\omega}),$$

  where $\left|H(e^{j\hat{\omega}})\right._{\hat{\omega}=\pi} \equiv 0$ independent of filter coefficients $h(n)$.

- The phase function and group delay function of a linear phase filter are independent of filter coefficients $h(n)$,

$$\angle H(e^{j\hat{\omega}}) = -\frac{N-1}{2}\hat{\omega} + \beta; \quad \tau(\hat{\omega}) = \frac{N-1}{2} \text{ where } \beta \text{ is a real constant value.}$$

Linear phase comb (Notch) filters:

- $h(n) = \delta(n) - \delta(n-L)$, (length $N = L+1$); $H(z) = 1 - z^{-L}$; and

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}L} = e^{-j\frac{\hat{\omega}L}{2}}(e^{j\frac{\hat{\omega}L}{2}} - e^{-j\frac{\hat{\omega}L}{2}}) = 2\sin(\frac{\hat{\omega}L}{2})e^{-j\frac{\hat{\omega}L}{2}+\frac{\pi}{2}}.$$

It is a linear phase filter with group delay $\tau(\hat{\omega}) = \frac{L}{2} = \frac{N-1}{2}$ and zeros (notches at) $\frac{\hat{\omega}L}{2} = 0$ or $l \cdot \pi$, $l = 0, 1, \cdots, L$, that is

$$\hat{\omega} = 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots, \frac{l \cdot 2\pi}{L}, \cdots, \pi. \quad \text{in real frequencies: } f = \frac{l \cdot f_s}{L}, \quad l = 0, 1, \cdots, \left\lfloor \frac{L}{2} \right\rfloor.$$

$h(n) = \delta(n) + \delta(n-L)$, (length $N = L+1$); $H(z) = 1 + z^{-L}$; and

$$H(e^{j\hat{\omega}}) = 1 + e^{-j\hat{\omega}L} = e^{-j\frac{\hat{\omega}L}{2}}(e^{j\frac{\hat{\omega}L}{2}} + e^{-j\frac{\hat{\omega}L}{2}}) = 2\cos(\frac{\hat{\omega}L}{2})e^{-j\frac{\hat{\omega}L}{2}}.$$

It is a linear phase filter with group delay $\tau(\hat{\omega}) = \frac{N-1}{2}$ and zeros (notches at) $\frac{\hat{\omega}L}{2} = \frac{l \cdot \pi}{2}$, $l = 1, 3, 5, \cdots$,odd integer $< 2L$, that is

$$\hat{\omega} = \frac{\pi}{L}, \frac{3\pi}{L}, \cdots, \frac{l \cdot \pi}{L}, \cdots, \pi. \quad \text{in real frequencies: } f = \frac{l \cdot f_s}{2L}, \quad l = 1, 3, 5, \cdots, \text{odd integer } \leq L.$$

Supplemental problems and computer project:

1. A LTI system is described by the following impulse response

$$h(n) = 2^{-n} \quad \text{for } n = -2, -1, \cdots, 3, 4$$

For a given input signal

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$x(n) =$	1	−1	1	2
$n =$	3	4	5	6

},

answer the following questions.

(a) Find the difference equation representation of the filter and system transfer function $B(z)$.

(b) What is the length of the zero–state response sequence $y(n)$?

(c) What are the beginning and ending indices of the zero–state response $y(n)$?

(d) Find some of the values of the zero–state response sequence

$$y(-2), y(0), y(4), y(8), y(11)$$

using the linear convolution sum.

(e) Verify your result using Mablab^{TM} function conv().

2. A linear time–invariant continuous–time integrator is described by the following differential equation.

$$y_a(t) = \frac{d^2 x_a(t)}{dt^2} - 5\frac{dx_a(t)}{dt} + x_a(t)$$

Find the corresponding difference equation with input $x(n)$, output $y(n)$ and $B(z)$ using the approximation,

$$\frac{dx_a(t)}{dt} \approx \frac{x_a(nT) - x_a(nT - T)}{T} = \frac{x(n) - x(n-1)}{T}.$$

Hint: $\frac{d^2 x_a(t)}{dt^2} = \frac{d}{dt}\{\frac{dx_a(t)}{dt}\}$.

3. Given the zeros of a FIR digital filter as follows: $z_0 = -1$, $z_{1,2} = e^{\pm j\frac{\pi}{4}}$.

   (a) Draw the pole/zero diagram.

   (b) Find the difference equation representation of the filter and system transfer function $B(z)$. Can you judge the filter is a linear phase filter or not from its filter coefficient $h(n)$? Is it?

   (c) Sketch the magnitude frequency response $\left|H(e^{j\hat{\omega}})\right|$ with the values clearly marked at $\hat{\omega} = 0, \frac{\pi}{2}$ and π.

   (d) Pick up a proper sampling frequency f_s such that the frequency components 200π and 800π in the following signal will be filtered out by the digital filter.

   $$x_a(t) = 0.2\sin(200\pi t + 1) - 0.3\cos(800\pi t - 1.5) + 2\cos(70\pi t - 1)$$

   and find the system steady state response $y_{ss}(n)$ to the signal.

   (e) Find the expression of the phase frequency response $\angle H(e^{j\hat{\omega}})$ and group delay $\tau(\hat{\omega})$.

   (f) Verify the results using Matlab^{TM}. You may use functions such as filter(), freqz(), poly(), fft(), abs(), angle(), impz() and zplane(), etc.

4. In the following signal

$$x_a(t) = e^{-0.5t} + 0.5\cos(400\pi t - 1) + 0.03\cos(1200\pi t - \frac{\pi}{2})$$

where $0 \le t \le 2$, the sinusoids are considered interferences to a low–pass type signal $e^{-0.5t}$. Design a Notch (Nulling) filter $B(z) = 1 + z^{-L}$ and a proper sampling frequency to eliminate the sinusoidal interferences.

   (a) Show the pole/zero diagram using zplane() function.

   (b) Plot frequency response (both magnitude and phase) of the filter in linear scale with frequency label in Hz. Plot the magnitude of the Fourier transform of $x(n)$ with frequency axis labeled in Hz, that is, k=0:N–1; f=k*fs/N; where N is the length of $x(n)$.

   (c) Using subplot() to show the plots of the original signal $x_a(t)$, filtered signal $y(n)$, and desired signal $e^{-0.5t}$.

   (d) Plot the magnitude of the Fourier transform of the filtered signal $y(n)$ with frequency axis labeled in Hz.

   (e) Draw conclusions from the observation.