

Chapter 7 and 8 supplemental notes

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@Apr. 2000

Z-transforms: Z-transform is the Laplace transform of a sampled sequence $x(n) = x_a(t)|_{t=nT}$ where $z = e^{sT}$ and $s = \sigma + j\omega$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-100)z^{100} + \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(100)z^{-100} + \cdots$$

for $z \in$ a Region of Convergence (R.O.C.). The R.O.C. of $X(z)$ is always bounded by poles of $X(z)$ and no poles of $X(z)$ are located within the R.O.C.

Type of sequences	R.O.C.
Right-sided $\sum_{n=n_0}^{\infty} x(n)z^{-n}$	$\begin{cases} z_0 < z , \text{ where } z_0 \text{ is the largest pole of } X(z) \text{ and } n_0 \geq 0 \\ z_0 < z < \infty, \text{ for } n_0 < 0 \end{cases}$
Left-sided $\sum_{n=-\infty}^{n_0} x(n)z^{-n}$	$\begin{cases} z < z_1 , \text{ where } z_1 \text{ is the smallest pole of } X(z) \text{ and } n_0 \leq 0 \\ 0 < z < z_1 , \text{ for } n_0 > 0 \end{cases}$
Two-sided $\sum_{n=-\infty}^{\infty} x(n)z^{-n}$	$ z_a < z < z_b $, there is no pole inside the annular shape of R.O.C.

Causality: The output $y(n)$ does not precede the input $x(n)$. For a causal LTI system, $h(n) = 0$ for $n < 0$. In terms of R.O.C., a system is causal, the R.O.C. of $H(z)$ must contain $z = \infty$, i.e., $|z_0| < |z|$.

Stability: A system is BIBO stable, if the output $y(n)$ is bounded for all the possible bounded inputs. For a LTI system, check (a) If $h(n)$ is absolutely summable, $\sum_{n=-\infty}^{\infty} |h(n)| = M < \infty$; or (b) If the R.O.C. of $H(z)$ contains the Unit Circle $e^{j\theta}$.

Common z-transforms and properties:

$$A\delta(n - n_0) \longleftrightarrow Az^{-n_0}, \quad \text{R.O.C. is } 0 < |z|, \text{ if } n_0 > 0; |z| < \infty, \text{ if } n_0 < 0$$

$$Au(n - n_0) \longleftrightarrow \frac{Az^{-n_0}}{1 - z^{-1}}, \quad \text{R.O.C. is } 1 < |z|, \text{ if } n_0 > 0; 1 < |z| < \infty, \text{ if } n_0 < 0$$

$$Aa^n u(n) \longleftrightarrow \frac{A}{1 - az^{-1}}, \quad \text{R.O.C. is } |a| < |z|$$

$$Ana^n u(n) \longleftrightarrow \frac{Aaz^{-1}}{(1 - az^{-1})^2}, \quad \text{R.O.C. is } |a| < |z|$$

$$Aa^n u(-n - 1) \longleftrightarrow \frac{-A}{1 - az^{-1}}, \quad \text{R.O.C. is } |z| < |a|$$

Shift Property: $x(n - n_0) \longleftrightarrow z^{-n_0} X(z)$, R.O.C. is R.O.C._X, but check $z = 0$ and ∞

Linear convolution: $x_1(n) * x_2(n) \longleftrightarrow X_1(z)X_2(z)$, R.O.C. is $\text{R.O.C.}_{X_1} \cap \text{R.O.C.}_{X_2}$, but check for pole/zero cancellations.

Multiplication by n : $nx(n) \longleftrightarrow -z \frac{dX(z)}{dz}$, R.O.C. is R.O.C._X , but check $z = \infty$

Inverse z-transforms:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^L b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{\sum_{k=0}^L b_k z^{N-k}}{z^N + \sum_{k=1}^N a_k z^{N-k}} = \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_L z^{N-L}}{z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N}$$

where usually $L < N$ for a proper function. If $L \geq N$, say $L - N = M$, use long division in ascending order of power

$$a_N z^{-N} + \dots + a_1 z^{-1} + a_0 \sqrt{b_L z^{-L} + \dots + b_1 z^{-1} + b_0}$$

to find

$$\begin{aligned} H(z) &= D(z) + H'(z) = D(z) + \frac{B'(z)}{A(z)} \\ &= d_M z^{-M} + d_{M-1} z^{-M+1} + \dots + d_1 z^{-1} + d_0 + \frac{\sum_{k=0}^{L-M-1} b'_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} \end{aligned}$$

where $H'(z)$ is a proper function. Partial fraction expansion of a proper function with simple poles

$$\frac{H(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_L z^{N-L-1}}{z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_L z^{N-L-1}}{\prod_{k=1}^N (z - p_k)} = \sum_{k=1}^N \frac{C_k}{(z - p_k)}$$

where $C_k = (z - p_k) \frac{H(z)}{z} \Big|_{z=p_k}$. Use R.O.C._H , $|z_a| < |z| < |z_b|$, to guide the table-lookup for each term in $H(z) = \sum_{k=1}^N \frac{C_k \cdot z}{(z - p_k)}$. If $|p_k| \leq |z_a| < |z|$, then the inverse z-transform of the term $\frac{C_k \cdot z}{(z - p_k)}$ is a right-sided sequence. If $|z| < |z_b| \leq |p_k|$, then the inverse z-transform of the term $\frac{C_k \cdot z}{(z - p_k)}$ is a left-sided sequence.

Long division method for numerical solution of inverse z-transform:

- Using long division in ascending order to generate left-sided sequence:

$$a_N z^{-N} + \dots + a_1 z^{-1} + a_0 \sqrt{b_L z^{-L} + \dots + b_1 z^{-1} + b_0}$$

- Using long division in descending order to generate right-sided sequence:

$$a_0 + a_1 z^{-1} + \dots + a_N z^{-N} \sqrt{b_0 + b_1 z^{-1} + \dots + b_L z^{-L}}$$

Sketch the magnitude frequency response from the pole/zero diagram:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^{N-L} \prod_{k=1}^L (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

where z_k and p_k are the zeros and poles of the system respectively.

$$\left|H(e^{j\hat{\omega}})\right| = \frac{|b_0| \prod_{k=1}^L |e^{j\hat{\omega}} - z_k|}{\prod_{k=1}^N |e^{j\hat{\omega}} - p_k|} = \frac{|b_0| \prod_{k=1}^L B_k}{\prod_{k=1}^N A_k}$$

where $B_k = |e^{j\hat{\omega}} - z_k|$ and $A_k = |e^{j\hat{\omega}} - p_k|$ are the distances from a frequency $0 \leq \hat{\omega} \leq \pi$ to zero z_k and pole p_k respectively. Note that $|H(1)|$ is the DC gain ($\hat{\omega} = 0$) and $|H(-1)|$ is the gain at $\hat{\omega} = \pi$.

Linear Phase FIR Filter and Its Implementations:

For a FIR filter, we can design the filter coefficients such that the filter has a desired magnitude frequency response (i.e. a desired frequency selectivity) and a linear phase frequency response (i.e. a constant group delay). This type of FIR filters are categorized as *linear phase FIR filter*. For a linear phase FIR filter, the unit impulse response sequence must satisfy the following conditions:

$$h(n) = h(N - 1 - n) \quad \text{even symmetry to } \frac{N-1}{2} \quad (1)$$

or

$$h(n) = -h(N - 1 - n) \quad \text{odd symmetry to } \frac{N-1}{2} \quad (2)$$

where integer N , that could be even or odd number, is the length of the filter.

For a linear phase filter, if z_0 is a zero of the filter, then z_0^{-1} must be also a zero of the filter. Since $h(n)$ is a real sequence, if z_0 is complex, then z_0^* must be a zero of $H(z)$, so as $(z_0^*)^{-1}$. This property can be proved as follows:

The system transfer function of a linear phase filter can be expressed as

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}. \quad (3)$$

Since $h(n) = \pm h(N - 1 - n)$, we can rewrite equation (3) to

$$H(z) = \sum_{n=0}^{N-1} \pm h(N - 1 - n)z^{-n}. \quad (4)$$

By substituting $l = N - 1 - n$, the above equation can be rewritten as

$$\begin{aligned} H(z) &= \sum_{l=N-1}^0 \pm h(l)z^{-(N-1-l)} \\ &= z^{-(N-1)} \sum_{l=0}^{N-1} \pm h(l)z^l \\ &= z^{-(N-1)} H(z^{-1}). \end{aligned} \quad (5)$$

If z_0 is a zero of a filter, then the transfer function $H(z)$ must satisfy

$$H(z)|_{z=z_0} = z_0^{-(N-1)} H(z_0^{-1}) = 0, \quad (6)$$

that is, z_0^{-1} must also be a zero of $H(z)$, due to $z_0^{-(N-1)} \neq 0$. The pole/zero diagram of a typical linear phase FIR filter with length $N = 9$ coefficients (8 zeros) is shown in figure(1).

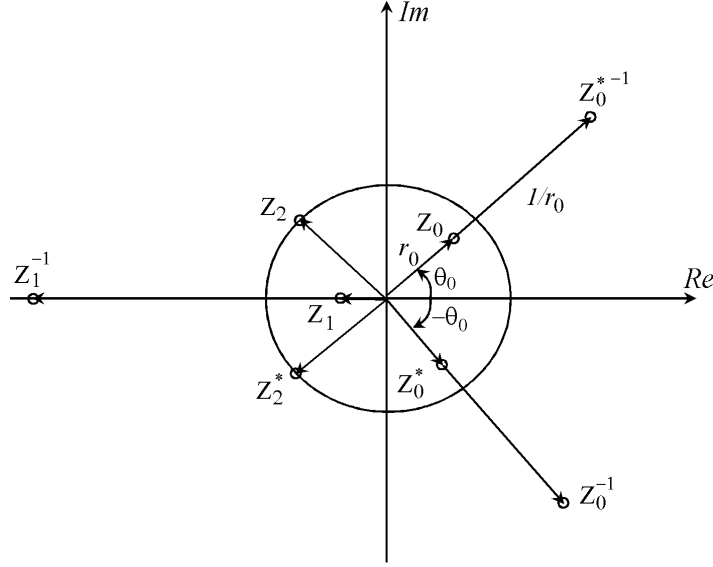


Figure 1: A typical zero diagram of a length $N = 9$ linear phase FIR filter

In many literatures, a linear phase filter with even symmetric or odd symmetric filter coefficients are referred to as TYPE I or TYPE III filters respectively, if the length N of the linear phase FIR filter is an odd number. On the other hand, it is referred to as TYPE II or TYPE IV filters respectively, if the length N of the linear phase FIR filter is an even number.

TYPE I or TYPE III filters: $h(n) = \pm h(N - 1 - n)$ and N is an odd integer number. We can rewrite the expression of the transfer function as follows.

$$H(z) = \sum_{n=0}^{(\frac{N-1}{2})-1} h(n)z^{-n} + h(\frac{N-1}{2})z^{-(\frac{N-1}{2})} + \quad (7)$$

$$\sum_{n=(\frac{N-1}{2})+1}^{N-1} h(n)z^{-n}$$

$$= \sum_{n=0}^{(\frac{N-1}{2})-1} h(n)z^{-n} + h(\frac{N-1}{2})z^{-(\frac{N-1}{2})} + \quad (8)$$

$$\sum_{l=0(\text{let } l=N-1-n)}^{(\frac{N-1}{2})-1} h(N-1-l)z^{-(N-1-l)}$$

$$= h(\frac{N-1}{2})z^{-(\frac{N-1}{2})} + \quad (9)$$

$$\sum_{n=0}^{(\frac{N-1}{2})-1} h(n) \cdot (z^{-n} \pm z^{-(N-1-n)}).$$

With the new expression which takes the advantage of the symmetric filter coefficients, an implementation with only $\frac{N-1}{2} + 1$ multiplications can be realized. Figure (2) shows the system diagram of a TYPE I or TYPE III filter.

TYPE II or TYPE IV filters: $h(n) = \pm h(N - 1 - n)$ and N is an even integer number. The transfer

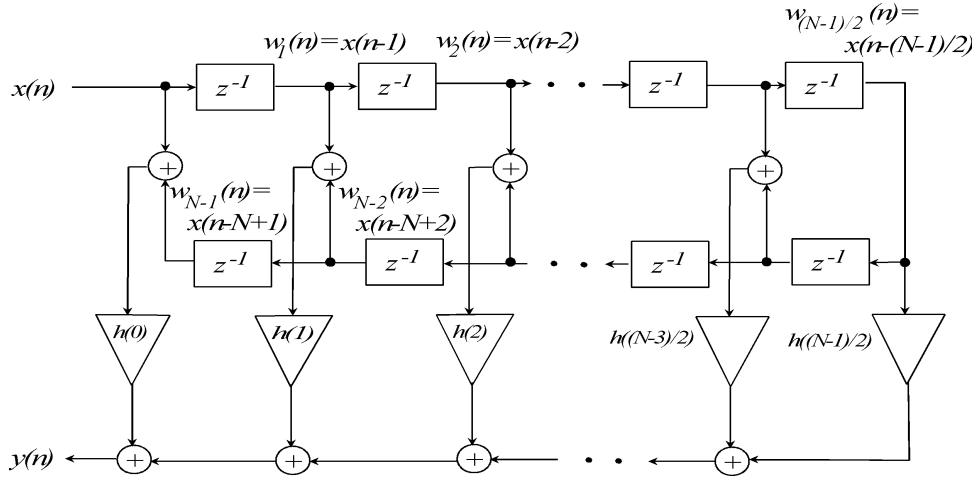


Figure 2: Direct realization of a TYPE I or TYPE III linear phase FIR filter

function can be rewritten as

$$H(z) = \sum_{n=0}^{(\frac{N}{2})-1} h(n)z^{-n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)z^{-n} \quad (10)$$

$$\begin{aligned}
&= \sum_{n=0}^{(\frac{N}{2})-1} h(n)z^{-n} + \\
&\quad \sum_{l=0(\text{let } l=N-1-n)}^{(\frac{N}{2})-1} h(N-1-l)z^{-(N-1-l)} \quad (11) \\
&= \sum_{n=0}^{(\frac{N}{2})-1} h(n) \cdot (z^{-n} \pm z^{-(N-1-n)}).
\end{aligned}$$

The system diagram of the TYPE II or TYPE IV filters is shown in figure (3). There are only $\frac{N}{2}$ multiplications in this implementation.

Laplace Transform to Z Transform through Sampling

Every real analog signal $x_a(t)$ has a limited bandwidth f_{\max} due to the finite signal energy. Therefore, every real analog signal has a Nyquist rate that is twice of the bandwidth of the signal, $f_N = 2f_{\max}$. According to the sampling theory, the analog signal can be sampled in to a discrete sequence without aliasing error by selecting a sampling frequency greater than the Nyquist rate, $f_s > f_N$.

To compare the Laplace transform of the analog signal and z -transform of its discrete counter part. We first set the model for the ideal sampling process:

$$\begin{aligned}
x_a(t) &= x(n) = x_a(t)|_{t=nT} \quad (12) \\
&= x_a(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT).
\end{aligned}$$

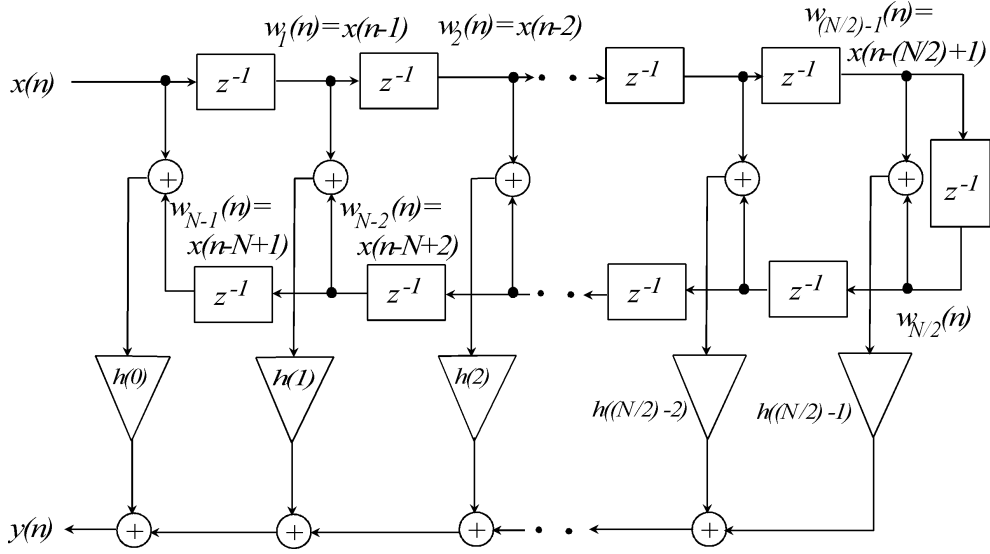


Figure 3: Direct realization of a TYPE II or TYPE IV linear phase FIR filter

The Laplace transforms of analog signal and its discrete counter part can be presented as:

$$X_a(s) = \int_{-\infty}^{\infty} x_a(t)e^{-st} dt \quad (13)$$

and

$$\begin{aligned} X_d(s) &= \int_{-\infty}^{\infty} x_a(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)e^{-st} dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_a(nT) \cdot \delta(t - nT)e^{-st} dt \\ &= \sum_{n=-\infty}^{\infty} x_a(nT) \cdot \int_{-\infty}^{\infty} \delta(t - nT)e^{-st} dt \\ &= \sum_{n=-\infty}^{\infty} x_a(nT) \cdot e^{-sTn} \\ &= \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = X(z) \end{aligned} \quad (14)$$

where

$$z = e^{sT} = e^{\sigma T} e^{j\omega T} \quad (15)$$

is a complex variable mapped from the complex variable $s = \sigma + j\omega$ of the s -domain through an exponential function. The Laplace transform of a discrete sequence is called z -transform and the complex variable z is in a domain called z -domain.

Equation (15) reveals the relationship between the Laplace transform of an analog signal and the

z -transform of its discrete counter part. The map between two domains can be summarized in table (16).

s -domain	z -domain
The frequency axis: $s = j\omega$	The Unit Circle: $z = e^{j\omega T} = e^{j\hat{\omega}}, \hat{\omega} = \omega T$ (radian)
The left-half of: $\sigma < 0$	Inside of the Unit Circle: $ z = e^{\sigma T} e^{j\omega} = e^{\sigma T} < 1$
The right-half of: $\sigma > 0$	Outside of the Unit Circle: $ z = e^{\sigma T} > 1$

(16)

For a continuous-time, causal and stable analog system $h_a(t)$, its Laplace transform $H_a(s)$ must have all the poles on the left-half s -domain, that is, the real part (σ) of all the poles must be less than 0. Therefore, the region of convergence (R.O.C.) of $H_a(s)$ is always greater than the real part of the largest pole(s) of $H_a(s)$ including $j\omega$ axis and $\sigma = \infty$, that is $\sigma_{\max_pole} < \sigma$. Its discrete-time counter part $H(z)$ must have all the poles inside the Unit circle according to the table (16) and the R.O.C. greater than the magnitude, $e^{\sigma_{\max_pole} T}$, of the largest pole(s) including the Unit Circle and ∞ . Another observation, if a continuous-time system is stable, then letting $s = j\omega$ yields

$$H_a(j\omega) = \int_{-\infty}^{\infty} h_a(t) e^{-j\omega t} dt. \quad (17)$$

This is the Fourier transform of the impulse response and represents the frequency response of the system. According to the table (16), for a stable discrete-time signal, the Fourier transform of the impulse response sequence $h(n)$ can be expressed as

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\hat{\omega} n} \quad (18)$$

that also represents the frequency response of the system.

In a discrete time situation, we also discuss non-causal systems and signals due to the easy implementation of delay in a computer. The stability of linear time-invariant (LTI) system can be generally stated as follows:

- A discrete-time LTI system is BIBO stable, if and only if, the region of convergence (R.O.C.) of $H(z)$ contains the Unit Circle.

Since the Laplace transform and z -transform are related by the sampling process with a sampling interval T , we would appreciate a process to convert the transfer function, $H_a(s)$, of a continuous-time system directly to $H(z)$ of a discrete-time system. The direct conversion from $H_a(s)$ to $H(z)$ is based on sampling process. Therefore, it is not suitable for an analog system $H_a(s)$ which is highpass in nature. The parameter of the direct conversion is the sampling interval T that must be selected according to the sampling theorem. The theory behind the direct conversion is the impulse-invariant as explained in figure (4). The conversion method:

- If $H_a(s)$ is proper and has N distinct poles, p_k , $k = 1, 2, \dots, N$, then the partial fraction expansion yields

$$H_a(s) = \sum_{k=1}^N \frac{A_k}{s - p_k}. \quad (19)$$

- Inverse Laplace transform yields

$$h_a(t) = \sum_{k=1}^N A_k \cdot e^{p_k \cdot t} u(t). \quad (20)$$

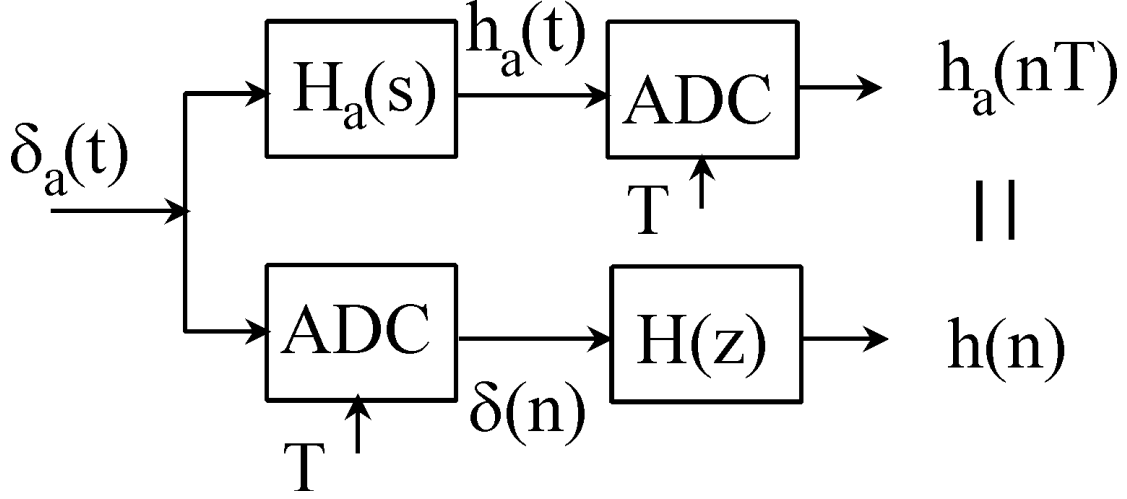


Figure 4: Impulse Invariant Concept: $y_a(n) = y(n)$

- Sampling the impulse response function yields

$$h_a(nT) = \sum_{k=1}^N A_k \cdot e^{p_k \cdot Tn} u(nT). \quad (21)$$

- According to the impulse invariant principle, $h_a(nT)$ must be equal to $h(n)$, we have

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^N [A_k \cdot e^{p_k \cdot Tn} u(nT)] z^{-n} \\ &= \sum_{k=1}^N A_k \left[\sum_{n=0}^{\infty} e^{p_k \cdot Tn} z^{-n} \right] \\ &= \sum_{k=1}^N \frac{A_k}{1 - e^{p_k \cdot T} z^{-1}} \end{aligned} \quad (22)$$

and R.O.C. is $\max_k \{ |e^{p_k \cdot T}| \} < |z|$. That is if the largest real part of the poles of $H_a(s)$ is σ_{\max} , then the R.O.C. is $e^{\sigma_{\max} T} < |z|$.

$$H_a(s) = \sum_{k=1}^N \frac{A_k}{s - p_k} \iff H(z) = \sum_{k=1}^N \frac{A_k}{1 - e^{p_k \cdot T} z^{-1}} \quad (23)$$

is the direct conversion of Laplace transform to z -transform via sampling process using the principle of impulse invariant. Note that the direct conversion only works for the lowpass type of systems.

Example 1 Determine $H(z)$ given that

$$H_a(s) = \frac{1}{s^2 + 3s + 2}.$$

Perform partial fraction expansion first:

$$H_a(s) = \frac{1}{s+1} + \frac{-1}{s+2}$$

The direct conversion yields:

$$H(z) = \frac{1}{1 - e^{-T}z^{-1}} - \frac{1}{1 - e^{-2T}z^{-1}}$$

Example 2 Determine $H(z)$ given that

$$H_a(s) = \frac{1}{s^2(s+1)}.$$

Perform partial fraction expansion first:

$$H_a(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

where

$$\begin{aligned} A &= s^2 H_a(s)|_{s=0} = 1 \\ B &= \frac{d}{ds}(s^2 H_a(s))|_{s=0} = -1 \\ C &= (s+1)H_a(s)|_{s=-1} = 1. \end{aligned}$$

Since inverse Laplace of $\frac{1}{s^2}$ and $\frac{1}{s}$ are $tu(t)$ and $u(t)$, the discrete-time sequences are $Tnu(n)$ and $u(n)$ respectively, the impulse invariant principle yields the direct conversion from Laplace to z -transform, that is,

$$H(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}$$

from $H_a(s) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$.

Example 3 Determine $H(z)$ given that

$$H_a(s) = \frac{s}{(s+2)^2}.$$

Perform partial fraction expansion first:

$$H_a(s) = \frac{A}{(s+2)^2} + \frac{B}{s+2}$$

where

$$\begin{aligned} A &= (s+2)^2 H_a(s)|_{s=-2} = -2 \\ B &= \frac{d}{ds}((s+2)^2 H_a(s))|_{s=-2} = 1. \end{aligned}$$

Since inverse Laplace of $\frac{1}{(s-a)^2}$ is $te^{at}u(t)$ and its discrete-time sequence is $nTe^{aTn}u(n)$, the impulse invariant principle yields

$$H(z) = \frac{-2Te^{-2T}z^{-1}}{(1-e^{-2T}z^{-1})^2} + \frac{1}{1-e^{-2T}z^{-1}}$$

from $H_a(s) = \frac{-2}{(s+2)^2} + \frac{1}{s+2}$.

The direct conversion table:

$H_s(s)$	$H(z)$
$\frac{1}{s}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s-a}$	$\frac{1}{1-e^{aT}z^{-1}}$
$\frac{1}{(s-a)^2}$	$\frac{Tz^{-1}}{(1-e^{aT}z^{-1})^2}$

(24)

The table can be used for most of simple causal analog systems in conjunction with partial fraction expansion of the transfer function $H_a(s)$.

The same impulse invariant principle can be realized by using complex convolution in the Laplace domain. The equation (12) can be expressed in Laplace transform (only causal systems are considered).

$$\begin{aligned}
 X_d(s) &= X_a(s) * \text{Laplace}\left\{\sum_{n=0}^{\infty} \delta(t - nT)\right\} \\
 &= X_a(s) * \int_0^{\infty} \sum_{n=0}^{\infty} \delta(t - nT) e^{-st} dt \\
 &= X_a(s) * \sum_{n=0}^{\infty} e^{-sTn} \\
 &= X_a(s) * \frac{1}{1 - e^{-sT}}
 \end{aligned}
 \tag{25}$$

where $e^{-\sigma T} < 1$ i.e. the R.O.C. is $\sigma > 0$. The complex convolution can be expressed as following,

$$X_d(s) = \int_C X_a(\lambda) \frac{1}{1 - e^{-(s-\lambda)T}} d\lambda$$

where C is a closed contour within the region of convergence in the λ domain. Using the fact that $1 = e^{j2\pi k}$, we can find that function

$$\frac{1}{1 - e^{-(s-\lambda)T}}$$

has infinite number of poles

$$\lambda_k = s + j\frac{2\pi}{T}k$$

on the right-half of λ domain due to the R.O.C. $\sigma > 0$ for all s . Therefore, the integral can be performed by selecting a circle enclosing all the poles of $X_a(\lambda)$ on the left-half of λ domain. The contour integral can be evaluated by the residues method:

$$X_d(s) = \sum_{\text{all poles of } X_a(\lambda)} \left\{ \text{Residues of } X_a(\lambda) \cdot \frac{1}{1 - e^{-(s-\lambda)T}} \right\}. \tag{26}$$

Note that $X(z) = X_d(s)$ with $z = e^{sT}$.

For a transfer function $X_a(s)$ that only has distinctive single poles:

$$X(z) = \left[\sum_k \left\{ (\lambda - p_k) \frac{X_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=p_k} \right]_{z=e^{sT}} \tag{27}$$

where p_k are the poles of $X_a(s)$.

For a transfer function that includes one m -multiple pole ($p_{k_m}^m$):

$$X(z) = \left[\sum_{k \neq k_m} \left\{ \frac{(\lambda - p_k) X_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=p_k} + \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \left\{ \frac{(\lambda - p_{k_m}^m)^m X_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=p_{k_m}^m} \right]_{z=e^{sT}} \quad (28)$$

Example 4 Determine $H(z)$ given that

$$H_a(s) = \frac{s}{(s+2)^2}.$$

$$\begin{aligned} H(z) &= \left[\frac{1}{(2-1)!} \frac{\partial^{2-1}}{\partial \lambda^{2-1}} \left\{ (\lambda+2)^2 \frac{H_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=-2} \right]_{z=e^{sT}} \\ &= \left[\frac{\partial}{\partial \lambda} \left\{ \frac{\lambda}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=-2} \right]_{z=e^{sT}} \\ &= \left[\left\{ \frac{1 - e^{-(s-\lambda)T} + \lambda T e^{-sT} e^{\lambda T}}{(1 - e^{-(s-\lambda)T})^2} \right\}_{\lambda=-2} \right]_{z=e^{sT}} \\ &= \left[\frac{1 - e^{-(s+2)T} - 2T e^{-sT} e^{-2T}}{(1 - e^{-(s+2)T})^2} \right]_{z=e^{sT}} \\ &= \frac{1 - e^{-2T} z^{-1} - 2T e^{-2T} z^{-1}}{(1 - e^{-2T} z^{-1})^2} \\ &= \frac{1}{1 - e^{-2T} z^{-1}} - \frac{2T e^{-2T} z^{-1}}{(1 - e^{-2T} z^{-1})^2}. \end{aligned}$$

It yields the same result as the other method.

Supplemental problems and computer project:

1. Determine $H(z)$ using the impulse invariant principle given that

$$H_a(s) = \frac{1}{s^2 + 5s + 6}.$$

2. Determine $H(z)$ using the impulse invariant principle given that

$$H_a(s) = \frac{s}{(s+2)(s+1)^2}.$$

3. Determine $H(z)$ using the impulse invariant principle given that

$$H_a(s) = \frac{3}{s^2(s+2)}.$$

4. An IIR filter has the following transfer function

$$H(z) = \frac{1 - 1.1z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}}.$$

- (a) Plot the pole-zero diagram.

- (b) Find all the possible R.O.C. for $H(z)$ and indicate all the R.O.C. on the pole-zero diagram.
- (c) Which one of the R.O.C. corresponding to a stable filter? Why?
- (d) Which one of the R.O.C. corresponding to a causal filter? Why?
- (e) Find the impulse response, $h(n)$, of the stable filter using inverse Z -transform
- (f) Use MatlabTM function `impz()` to generate first 10 values of the impulse response and compare them with $h(n)$ that you find using inverse Z transform.
- (g) Sketch the magnitude frequency response of the filter according to its pole-zero diagram. What kind of frequency selective filter is this?
- (h) Calculate the steady state response to a sinusoidal signal which has a frequency equal to $3/8$ of the sampling frequency.
- (i) For an input signal $x(n) = 2(0.5)^n u(n)$ and initial conditions $y(-1) = 1$, $y(0) = -0.5$, calculate the first 10 values of $y(n)$ iteratively by hand.
- (j) Now for the same input signal and same initial conditions, calculate the first 10 values of $y(n)$ by using MatlabTM functions `filtic()` and `filter()`.