Chapter 7 and 8 supplemental notes

Instructor: Kefu Xue, Ph. D., Dept. of EE, WSU

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Z–transforms: Z–transform is the Laplace transform of a sampled sequence $x(n) = x_a(t)|_{t=nT}$ where $z = e^{sT}$ and $s = \sigma + j\omega$.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-100)z^{100} + \cdots + x(-2)z^2 + x(-1)z + x(0) + \\ & \quad x(1)z^{-1} + x(2)z^{-2} + \cdots + x(100)z^{-100} + \cdots \end{aligned}$$

for $z \in$ a Region of Convergence (R.O.C.). The R.O.C. of $X(z)$ is always bounded by poles of $X(z)$ and no poles of $X(z)$ are located within the R.O.C.

Type of sequences	R.O.C.
Right–sided $\sum_{n=n_0}^{\infty} x(n)z^{-n}$	$\begin{cases} \|z_0\| < \|z\|, \text{ where } z_0 \text{ is the largest pole of } X(z) \text{ and } n_0 \geq 0 \\ \|z_0\| < \|z\| < \infty, \text{ for } n_0 < 0 \end{cases}$
Left–sided $\sum_{n=-\infty}^{n_0} x(n)z^{-n}$	$\begin{cases} \|z\| < \|z_1\|, \text{ where } z_1 \text{ is the smallest pole of } X(z) \text{ and } n_0 \leq 0 \\ 0 < \|z\| < \|z_1\|, \text{ for } n_0 > 0 \end{cases}$
Two-sided $\sum_{n=-\infty}^{\infty} x(n)z^{-n}$	$\|z_a\| < \|z\| < \|z_b\|$, there is no pole inside the annular shape of R.O.C.

Causality: The output $y(n)$ does not precede the input $x(n)$. For a causal LTI system, $h(n) = 0$ for $n < 0$. In terms of R.O.C., a system is causal, the R.O.C. of $H(z)$ must contain $z = \infty$, i.e., $|z_0| < |z|$.
Stability: A system is BIBO stable, if the output $y(n)$ is bounded for all the possible bounded inputs. For a LTI system, check (a) If $h(n)$ is absolutely summable, $\sum_{n=-\infty}^{\infty} |h(n)| = M < \infty$; or (b) If the R.O.C. of $H(z)$ contains the Unit Circle $e^{j\theta}$.
Common z–transforms and properties:

$$A\delta(n-n_0) \longleftrightarrow Az^{-n_0}, \quad \text{R.O.C. is } 0 < |z|, \text{ if } n_0 > 0;\ |z| < \infty, \text{ if } n_0 < 0$$

$$Au(n-n_0) \longleftrightarrow \frac{Az^{-n_0}}{1-z^{-1}}, \quad \text{R.O.C. is } 1 < |z|, \text{ if } n_0 > 0;\ 1 < |z| < \infty, \text{ if } n_0 < 0$$

$$Aa^n u(n) \longleftrightarrow \frac{A}{1-az^{-1}}, \quad \text{R.O.C. is } |a| < |z|$$

$$Ana^n u(n) \longleftrightarrow \frac{Aaz^{-1}}{(1-az^{-1})^2}, \quad \text{R.O.C. is } |a| < |z|$$

$$Aa^n u(-n-1) \longleftrightarrow \frac{-A}{1-az^{-1}}, \quad \text{R.O.C. is } |z| < |a|$$

Shift Property: $x(n-n_0) \longleftrightarrow z^{-n_0}X(z)$, R.O.C. is R.O.C.$_X$, but check $z = 0$ and ∞

Linear convolution: $x_1(n) * x_2(n) \longleftrightarrow X_1(z)X_2(z)$, R.O.C. is R.O.C.$_{X_1} \cap$ R.O.C.$_{X_2}$, but check for pole/zero cancellations.

Multiplication by n: $nx(n) \longleftrightarrow -z\frac{dX(z)}{dz}$, R.O.C. is R.O.C.$_X$, but check $z = \infty$

Inverse z–transforms:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{L} b_k z^{N-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_L z^{N-L}}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

where usually $L < N$ for a proper function. If $L \geq N$, say $L - N = M$, use long division in ascending order of power

$$a_N z^{-N} + \cdots + a_1 z^{-1} + a_0 \overline{\big) b_L z^{-L} + \cdots + b_1 z^{-1} + b_0}$$

to find

$$\begin{aligned} H(z) &= D(z) + H'(z) = D(z) + \frac{B'(z)}{A(z)} \\ &= d_M z^{-M} + d_{M-1} z^{-M+1} + \cdots + d_1 z^{-1} + d_0 + \frac{\sum_{k=0}^{L-M-1} b'_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \end{aligned}$$

where $H'(z)$ is a proper function. Partial fraction expansion of a proper function with simple poles

$$\frac{H(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_L z^{N-L-1}}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_L z^{N-L-1}}{\prod_{k=1}^{N} (z - p_k)} = \sum_{k=1}^{N} \frac{C_k}{(z - p_k)}$$

where $C_k = (z - p_k)\frac{H(z)}{z}|_{z=p_k}$. Use R.O.C.$_H$, $|z_a| < |z| < |z_b|$, to guide the table–lookup for each term in $H(z) = \sum_{k=1}^{N} \frac{C_k \cdot z}{(z-p_k)}$. If $|p_k| \leq |z_a| < |z|$, then the inverse z–transform of the term $\frac{C_k \cdot z}{(z-p_k)}$ is a right-sided sequence. If $|z| < |z_b| \leq |p_k|$, then the inverse z–transform of the term $\frac{C_k \cdot z}{(z-p_k)}$ is a left-sided sequence.

Long division method for numerical solution of inverse z–transform:

- Using long division in ascending order to generate left–sided sequence:

$$a_N z^{-N} + \cdots + a_1 z^{-1} + a_0 \overline{\big) b_L z^{-L} + \cdots + b_1 z^{-1} + b_0}$$

- Using long division in descending order to generate right–sided sequence:

$$a_0 + a_1 z^{-1} + \cdots + a_N z^{-N} \overline{\big) b_0 + b_1 z^{-1} + \cdots + b_L z^{-L}}$$

Sketch the magnitude frequency response from the pole/zero diagram:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^{N-L} \prod_{k=1}^{L} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

where z_k and p_k are the zeros and poles of the system respectively.

$$\left|H(e^{j\hat{\omega}})\right| = \frac{|b_0| \prod_{k=1}^{L} \left|e^{j\hat{\omega}} - z_k\right|}{\prod_{k=1}^{N} \left|e^{j\hat{\omega}} - p_k\right|} = \frac{|b_0| \prod_{k=1}^{L} B_k}{\prod_{k=1}^{N} A_k}$$

where $B_k = \left|e^{j\hat{\omega}} - z_k\right|$ and $A_k = \left|e^{j\hat{\omega}} - p_k\right|$ are the distances from a frequency $0 \leq \hat{\omega} \leq \pi$ to zero z_k and pole p_k respectively. Note that $|H(1)|$ is the DC gain ($\hat{\omega} = 0$) and $|H(-1)|$ is the gain at $\hat{\omega} = \pi$.

Linear Phase FIR Filter and Its Implementations:

For a FIR filter, we can design the filter coefficients such that the filter has a desired magnitude frequency response (i.e. a desired frequency selectivity) and a linear phase frequency response (i.e. a constant group delay). This type of FIR filters are categorized as *linear phase FIR filter*. For a linear phase FIR filter, the unit impulse response sequence must satisfy the following conditions:

$$h(n) = h(N-1-n) \qquad \text{even symmetry to } \frac{N-1}{2} \tag{1}$$

or

$$h(n) = -h(N-1-n) \qquad \text{odd symmetry to } \frac{N-1}{2} \tag{2}$$

where integer N, that could be even or odd number, is the length of the filter.

For a linear phase filter, if z_0 is a zero of the filter, then z_0^{-1} must be also a zero of the filter. Since $h(n)$ is a real sequence, if z_0 is complex, then z_0^* must be a zero of $H(z)$, so as $(z_0^*)^{-1}$. This property can be proved as follows:

The system transfer function of a linear phase filter can be expressed as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}. \tag{3}$$

Since $h(n) = \pm h(N-1-n)$, we can rewrite equation (3) to

$$H(z) = \sum_{n=0}^{N-1} \pm h(N-1-n) z^{-n}. \tag{4}$$

By substituting $l = N-1-n$, the above equation can be rewritten as

$$\begin{aligned} H(z) &= \sum_{l=N-1}^{0} \pm h(l) z^{-(N-1-l)} \\ &= z^{-(N-1)} \sum_{l=0}^{N-1} \pm h(l) z^{l} \\ &= z^{-(N-1)} H(z^{-1}). \end{aligned} \tag{5}$$

If z_0 is a zero of a filter, then the transfer function $H(z)$ must satisfy

$$H(z)\big|_{z=z_0} = z_0^{-(N-1)} H(z_0^{-1}) = 0, \tag{6}$$

that is, z_0^{-1} must also be a zero of $H(z)$, due to $z_0^{-(N-1)} \neq 0$. The pole/zero diagram of a typical linear phase FIR filter with length $N = 9$ coefficients (8 zeros) is shown in figure(1).

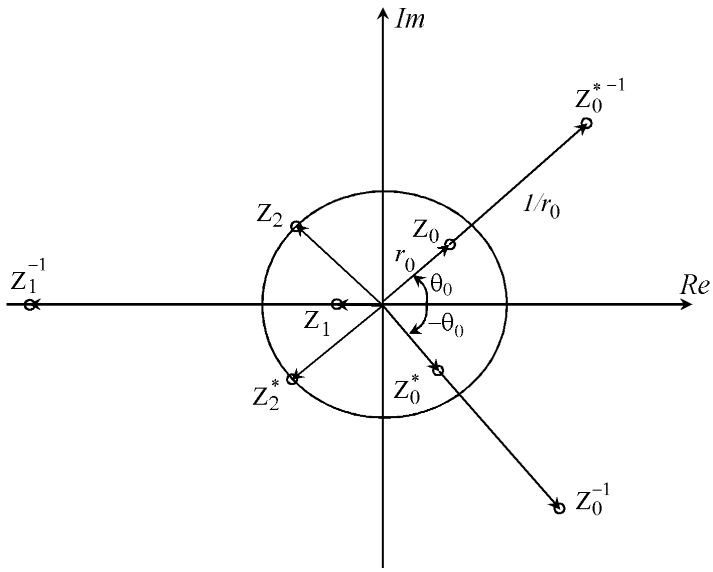

Figure 1: A typical zero diagram of a length $N = 9$ linear phase FIR filter

In many literatures, a linear phase filter with even symmetric or odd symmetric filter coefficients are referred to as TYPE I or TYPE III filters respectively, if the length N of the linear phase FIR filter is an odd number. On the other hand, it is referred to as TYPE II or TYPE IV filters respectively, if the length N of the linear phase FIR filter is an even number.

TYPE I or TYPE III filters: $h(n) = \pm h(N-1-n)$ and N is an odd integer number. We can rewrite the expression of the transfer function as follows.

$$\begin{aligned} H(z) &= \sum_{n=0}^{(\frac{N-1}{2})-1} h(n)z^{-n} + h(\frac{N-1}{2})z^{-(\frac{N-1}{2})} + \qquad (7) \\ &\quad \sum_{n=(\frac{N-1}{2})+1}^{N-1} h(n)z^{-n} \\ &= \sum_{n=0}^{(\frac{N-1}{2})-1} h(n)z^{-n} + h(\frac{N-1}{2})z^{-(\frac{N-1}{2})} + \\ &\quad \sum_{l=0(\text{let } l=N-1-n)}^{(\frac{N-1}{2})-1} h(N-1-l)z^{-(N-1-l)} \qquad (8) \\ &= h(\frac{N-1}{2})z^{-(\frac{N-1}{2})} + \\ &\quad \sum_{n=0}^{(\frac{N-1}{2})-1} h(n)\cdot(z^{-n} \pm z^{-(N-1-n)}). \qquad (9) \end{aligned}$$

With the new expression which takes the advantage of the symmetric filter coefficients, an implementation with only $\frac{N-1}{2}+1$ multiplications can be realized. Figure (2) shows the system diagram of a TYPE I or TYPE III filter.

TYPE II or TYPE IV filters: $h(n) = \pm h(N-1-n)$ and N is an even integer number. The transfer

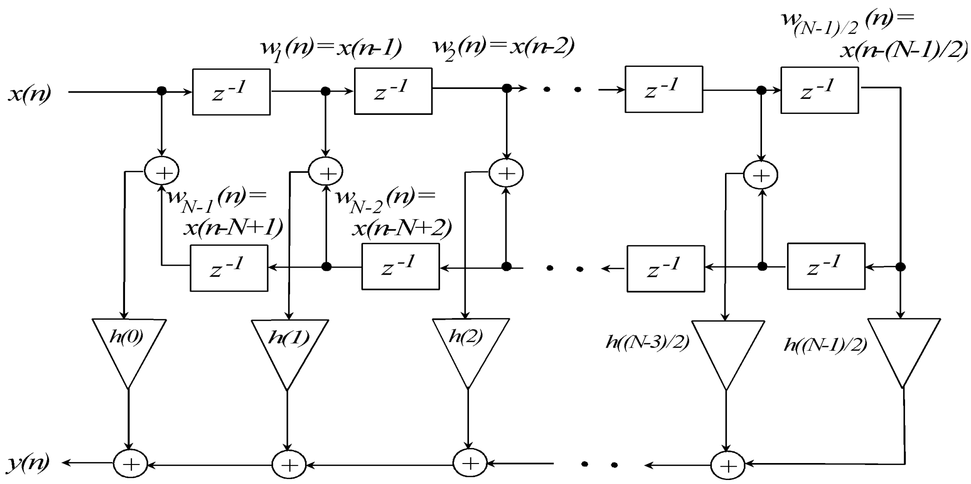

Figure 2: Direct realization of a TYPE I or TYPE III linear phase FIR filter

function can be rewritten as

$$H(z) = \sum_{n=0}^{(\frac{N}{2})-1} h(n)z^{-n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)z^{-n} \tag{10}$$

$$\begin{aligned} &= \sum_{n=0}^{(\frac{N}{2})-1} h(n)z^{-n} + \\ &\sum_{l=0 (\text{let } l=N-1-n)}^{(\frac{N}{2})-1} h(N-1-l)z^{-(N-1-l)} \end{aligned} \tag{11}$$

$$= \sum_{n=0}^{(\frac{N}{2})-1} h(n) \cdot (z^{-n} \pm z^{-(N-1-n)}).$$

The system diagram of the TYPE II or TYPE IV filters is shown in figure (3). There are only $\frac{N}{2}$ multiplications in this implementation.

Laplace Transform to Z Transform through Sampling

Every real analog signal $x_a(t)$ has a limited bandwidth $f_{\max}$ due to the finite signal energy. Therefore, every real analog signal has a Nyquest rate that is twice of the bandwidth of the signal, $f_N = 2f_{\max}$. According to the sampling theory, the analog signal can be sampled in to a discrete sequence without aliasing error by selecting a sampling frequency greater than the Nyquest rate, $f_s > f_N$.

To compare the Laplace transform of the analog signal and z–transform of its discrete counter part. We first set the model for the ideal sampling process:

$$\begin{aligned} x_d(t) &= x(n) = x_a(t)|_{t=nT} \\ &= x_a(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT). \end{aligned} \tag{12}$$

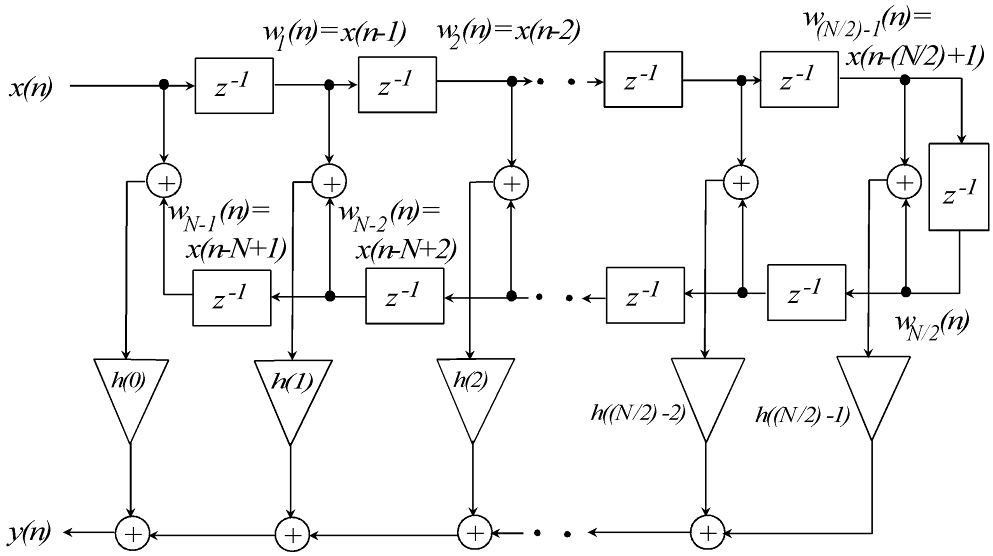

Figure 3: Direct realization of a TYPE II or TYPE IV linear phase FIR filter

The Laplace transforms of analog signal and its discrete counter part can be presented as:

$$X_a(s) = \int_{-\infty}^{\infty} x_a(t) e^{-st} dt \tag{13}$$

and

$$\begin{aligned} X_d(s) &= \int_{-\infty}^{\infty} x_a(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_a(nT) \cdot \delta(t-nT) e^{-st} dt \\ &= \sum_{n=-\infty}^{\infty} x_a(nT) \cdot \int_{-\infty}^{\infty} \delta(t-nT) e^{-st} dt \\ &= \sum_{n=-\infty}^{\infty} x_a(nT) \cdot e^{-sTn} \\ &= \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = X(z) \end{aligned} \tag{14}$$

where

$$z = e^{sT} = e^{\sigma T} e^{j\omega} \tag{15}$$

is a complex variable mapped from the complex variable $s = \sigma + j\omega$ of the s–domain through an exponential function. The Laplace transform of a discrete sequence is called z–transform and the complex variable z is in a domain called z–domain.

Equation (15) reveals the relationship between the Laplace transform of an analog signal and the

z–transform of its discrete counter part. The map between two domains can be summarized in table (16).

s–domain	z–domain
The frequency axis: $s = j\omega$	The Unit Circle: $z = e^{j\omega T} = e^{j\hat{\omega}}, \hat{\omega} = \omega T$ (radian)
The left–half of: $\sigma < 0$	Inside of the Unit Circle: $\|z\| = \|e^{\sigma T} e^{j\omega}\| = e^{\sigma T} < 1$
The right–half of: $\sigma > 0$	Outside of the Unit Circle: $\|z\| = e^{\sigma T} > 1$

(16)

For a continuous–time, causal and stable analog system $h_a(t)$, its Laplace transform $H_a(s)$ must have all the poles on the left–half s–domain, that is, the real part (σ) of all the poles must be less than 0. Therefore, the region of convergence (R.O.C.) of $H_a(s)$ is always greater than the real part of the largest pole(s) of $H_a(s)$ including $j\omega$ axis and $\sigma = \infty$, that is $\sigma_{\max\_pole} < \sigma$. Its discrete–time counter part $H(z)$ must have all the poles inside the Unit circle according to the table (16) and the R.O.C. greater than the magnitude, $e^{\sigma_{\max\_pole} T}$, of the largest pole(s) including the Unit Circle and ∞. Another observation, if a continuous–time system is stable, then letting $s = j\omega$ yields

$$H_a(j\omega) = \int_{-\infty}^{\infty} h_a(t) e^{-j\omega t} dt. \tag{17}$$

This is the Fourier transform of the impulse response and represents the frequency response of the system. According to the table (16), for a stable discrete–time signal, the Fourier transform of the impulse response sequence $h(n)$ can be expressed as

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\hat{\omega}} \tag{18}$$

that also represents the frequency response of the system.

In a discrete time situation, we also discuss non–causal systems and signals due to the easy implementation of delay in a computer. The stability of linear time–invariant (LTI) system can be generally stated as follows:

- A discrete–time LTI system is BIBO stable, if and only if, the region of convergence (R.O.C.) of $H(z)$ contains the Unit Circle.

Since the Laplace transform and z–transform are related by the sampling process with a sampling interval T, we would appreciate a process to convert the transfer function, $H_a(s)$, of a continuous–time system directly to $H(z)$ of a discrete–time system. The direct conversion from $H_a(s)$ to $H(z)$ is based on sampling process. Therefore, it is not suitable for an analog system $H_a(s)$ which is highpass in nature. The parameter of the direct conversion is the sampling interval T that must be selected according to the sampling theorem. The theory behind the direct conversion is the impulse–invariant as explained in figure (4). The conversion method:

- If $H_a(s)$ is proper and has N distinct poles, $p_k,\ k = 1, 2, \cdots, N$, then the partial fraction expansion yields

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - p_k}. \tag{19}$$

- Inverse Laplace transform yields

$$h_a(t) = \sum_{k=1}^{N} A_k \cdot e^{p_k \cdot t} u(t). \tag{20}$$

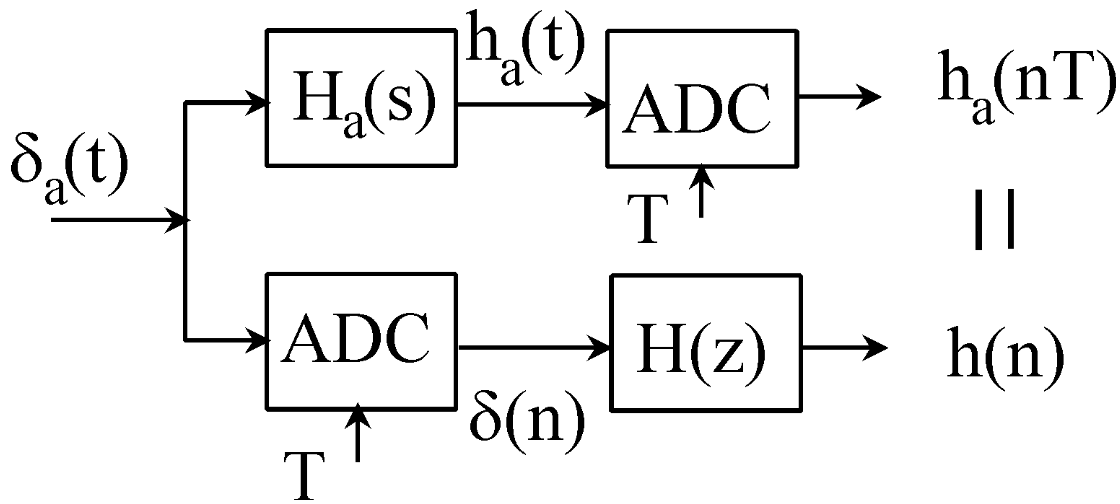

Figure 4: Impulse Invariant Concept: $y_a(n) = y(n)$

- Sampling the impulse response function yields

$$h_a(nT) = \sum_{k=1}^{N} A_k \cdot e^{p_k \cdot Tn} u(nT). \tag{21}$$

- According to the impulse invariant principle, $h_a(nT)$ must be equal to $h(n)$, we have

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{N} [A_k \cdot e^{p_k \cdot Tn} u(nT)] z^{-n} \\ &= \sum_{k=1}^{N} A_k [\sum_{n=0}^{\infty} e^{p_k \cdot Tn} z^{-n}] \\ &= \sum_{k=1}^{N} \frac{A_k}{1 - e^{p_k \cdot T} z^{-1}} \end{aligned} \tag{22}$$

and R.O.C. is $\max_k \left\{ |e^{p_k \cdot T}| \right\} < |z|$. That is if the largest real part of the poles of $H_a(s)$ is $\sigma_{\max}$, then the R.O.C. is $e^{\sigma_{\max} T} < |z|$.

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - p_k} \Longleftrightarrow H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{p_k \cdot T} z^{-1}} \tag{23}$$

is the direct conversion of Laplace transform to z–transform via sampling process using the principle of impulse invariant. Note that the direct conversion only works for the lowpass type of systems.

Example 1 *Determine $H(z)$ given that*

$$H_a(s) = \frac{1}{s^2 + 3s + 2}.$$

Perform partial fraction expansion first:

$$H_a(s) = \frac{1}{s+1} + \frac{-1}{s+2}$$

The direct conversion yields:

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}}$$

Example 2 *Determine $H(z)$ given that*

$$H_a(s) = \frac{1}{s^2(s+1)}.$$

Perform partial fraction expansion first:

$$H_a(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

where

$$\begin{aligned} A &= s^2 H_a(s)|_{s=0} = 1 \\ B &= \frac{d}{ds}(s^2 H_a(s))|_{s=0} = -1 \\ C &= (s+1)H_a(s)|_{s=-1} = 1. \end{aligned}$$

Since inverse Laplace of $\frac{1}{s^2}$ and $\frac{1}{s}$ are $tu(t)$ and $u(t)$, the discrete–time sequences are $Tnu(n)$ and $u(n)$ respectively, the impulse invariant principle yields the direct conversion from Laplace to z–transform, that is,

$$H(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}$$

from $H_a(s) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$.

Example 3 *Determine $H(z)$ given that*

$$H_a(s) = \frac{s}{(s+2)^2}.$$

Perform partial fraction expansion first:

$$H_a(s) = \frac{A}{(s+2)^2} + \frac{B}{s+2}$$

where

$$\begin{aligned} A &= (s+2)^2 H_a(s)|_{s=-2} = -2 \\ B &= \frac{d}{ds}((s+2)^2 H_a(s))|_{s=-2} = 1. \end{aligned}$$

Since inverse Laplace of $\frac{1}{(s-a)^2}$ is $te^{at}u(t)$ and its discrete–time sequence is $nTe^{aTn}u(n)$, the impulse invariant principle yields

$$H(z) = \frac{-2Te^{-2T}z^{-1}}{(1-e^{-2T}z^{-1})^2} + \frac{1}{1-e^{-2T}z^{-1}}$$

from $H_a(s) = \frac{-2}{(s+2)^2} + \frac{1}{s+2}$.

The direct conversion table:

$H_s(s)$	$H(z)$
$\frac{1}{s}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s-a}$	$\frac{1}{1-e^{aT}z^{-1}}$
$\frac{1}{(s-a)^2}$	$\frac{Te^{aT}z^{-1}}{(1-e^{aT}z^{-1})^2}$

(24)

The table can be used for most of simple causal analog systems in conjunction with partial fraction expansion of the transfer function $H_a(s)$.

The same impulse invariant principle can be realized by using complex convolution in the Laplace domain. The equation (12) can be expressed in Laplace transform (only causal systems are considered).

$$\begin{aligned} X_d(s) &= X_a(s) * \text{Laplace}\{\sum_{n=0}^{\infty} \delta(t-nT)\} \\ &= X_a(s) * \int_0^{\infty} \sum_{n=0}^{\infty} \delta(t-nT) e^{-st} dt \\ &= X_a(s) * \sum_{n=0}^{\infty} e^{-sTn} \\ &= X_a(s) * \frac{1}{1-e^{-sT}} \end{aligned} \tag{25}$$

where $e^{-\sigma T} < 1$ i.e. the R.O.C. is $\sigma > 0$. The complex convolution can be expressed as following,

$$X_d(s) = \int_C X_a(\lambda) \frac{1}{1-e^{-(s-\lambda)T}} d\lambda$$

where C is a closed contour within the region of convergence in the λ domain. Using the fact that $1 = e^{j2\pi k}$, we can find that function

$$\frac{1}{1-e^{-(s-\lambda)T}}$$

has infinite number of poles

$$\lambda_k = s + j\frac{2\pi}{T}k$$

on the right–half of λ domain due to the R.O.C. $\sigma > 0$ for all s. Therefore, the integral can be performed by selecting a circle enclosing all the poles of $X_a(\lambda)$ on the left–half of λ domain. The contour integral can be evaluated by the residues method:

$$X_d(s) = \sum_{\text{all poles of } X_a(\lambda)} \left\{ \text{Residues of } X_a(\lambda) \cdot \frac{1}{1-e^{-(s-\lambda)T}} \right\}. \tag{26}$$

Note that $X(z) = X_d(s)$ with $z = e^{sT}$.

For a transfer function $X_a(s)$ that only has distinctive single poles:

$$X(z) = \left[\sum_k \left\{ (\lambda - p_k) \frac{X_a(\lambda)}{1-e^{-(s-\lambda)T}} \right\}_{\lambda = p_k} \right]_{z=e^{sT}} \tag{27}$$

where p_k are the poles of $X_a(s)$.

For a transfer function that includes one m–multiple pole ($p_{k_m}^m$):

$$X(z) = \left[\sum_{k \neq k_m} \left\{ \frac{(\lambda - p_k) X_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda = p_k} + \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \left\{ \frac{(\lambda - p_{k_m}^m)^m X_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda = p_{k_m}^m} \right]_{z = e^{sT}} \tag{28}$$

Example 4 *Determine $H(z)$ given that*

$$H_a(s) = \frac{s}{(s+2)^2}.$$

$$\begin{aligned}
H(z) &= \left[\frac{1}{(2-1)!} \frac{\partial^{2-1}}{\partial \lambda^{2-1}} \left\{ (\lambda+2)^2 \frac{H_a(\lambda)}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=-2} \right]_{z=e^{sT}} \\
&= \left[\frac{\partial}{\partial \lambda} \left\{ \frac{\lambda}{1 - e^{-(s-\lambda)T}} \right\}_{\lambda=-2} \right]_{z=e^{sT}} \\
&= \left[\left\{ \frac{1 - e^{-(s-\lambda)T} + \lambda T e^{-sT} e^{\lambda T}}{(1 - e^{-(s-\lambda)T})^2} \right\}_{\lambda=-2} \right]_{z=e^{sT}} \\
&= \left[\frac{1 - e^{-(s+2)T} - 2T e^{-sT} e^{-2T}}{(1 - e^{-(s+2)T})^2} \right]_{z=e^{sT}} \\
&= \frac{1 - e^{-2T} z^{-1} - 2T e^{-2T} z^{-1}}{(1 - e^{-2T} z^{-1})^2} \\
&= \frac{1}{1 - e^{-2T} z^{-1}} - \frac{2T e^{-2T} z^{-1}}{(1 - e^{-2T} z^{-1})^2}.
\end{aligned}$$

It yields the same result as the other method.

Supplemental problems and computer project:

1. Determine $H(z)$ using the impulse invariant principle given that

$$H_a(s) = \frac{1}{s^2 + 5s + 6}.$$

2. Determine $H(z)$ using the impulse invariant principle given that

$$H_a(s) = \frac{s}{(s+2)(s+1)^2}.$$

3. Determine $H(z)$ using the impulse invariant principle given that

$$H_a(s) = \frac{3}{s^2(s+2)}.$$

4. An IIR filter has the following transfer function

$$H(z) = \frac{1 - 1.1z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}}.$$

   (a) Plot the pole–zero diagram.

(b) Find all the possible R.O.C. for $H(z)$ and indicate all the R.O.C. on the pole–zero diagram.

(c) Which one of the R.O.C. corresponding to a stable filter? Why?

(d) Which one of the R.O.C. corresponding to a causal filter? Why?

(e) Find the impulse response, $h(n)$, of the stable filter using inverse Z–transform

(f) Use Matlab^{TM} function impz() to generate first 10 values of the impulse response and compare them with $h(n)$ that you find using inverse Z transform.

(g) Sketch the magnitude frequency response of the filter according to its pole–zero diagram. What kind of frequency selective filter is this?

(h) Calculate the steady state response to a sinusoidal signal which has a frequency equal to 3/8 of the sampling frequency.

(i) For an input signal $x(n) = 2(0.5)^n u(n)$ and initial conditions $y(-1) = 1$, $y(0) = -0.5$, calculate the first 10 values of $y(n)$ iteratively by hand.

(j) Now for the same input signal and same initial conditions, calculate the first 10 values of $y(n)$ by using Matlab^{TM} functions filtic() and filter().