EE322/EE522
Digital Signals and Systems

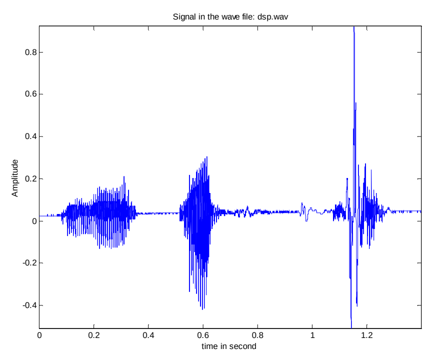
1.01: Signals and Signal Models

@ Kefu Xue, Ph.D., June 2006

Media, Sensors and Real Signals

- Air Pressure, Microphone, Speaker, Audio signals: Speech and Music
- Light, Camera, LCD display, video/image signals: Movies and Pictures
- Electrical Field, Electrodes, ECG & EEG signals: (electrocardiogram and electroencephalography)
- Liquid and Gas Pressure, Piezoelectric transducer, ultrasonic signals: Medical image, sensing objects and measuring distances in robotic and industrial applications.

A Recorded Sound Signal: "DSP" wave file:



Matlab --- a tool used extensively in EE Curriculum

```
% EE322 Example
%
% @ Kefu Xue, Ph.D., June 2006
%
filnm=input('Enter the wave file name: ','s');
[x,fs,bits]=wavread(filnm);
N=length(x);
t=[0:N-1]/fs;
figure;plot(t,x);axis tight;xlabel('time in second');
ylabel('Amplitude');
title(['Signal in the wave file: ' filnm]);
%
% end of file
```

Signals: Energy and Amplitude Variations

- All signals consume energy.
- All signals vary their amplitude with respect to time, space or both.
- Signals are often represented mathematically as a function $x_a(t)$.
- Continuous time signal: the domain t is a continuous independent variable.
- Discrete time signal: the domain t is a discrete independent variable
- Digital signal: Both the domain t and range x( ) are discrete variables.

Typical continuous time signals and their mathematical representations:

Unit Step function: $u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$

Unit Impulse function: $\delta(t) = \begin{cases} \infty \; (or \; \frac{1}{\varepsilon}) & t = 0 \\ 0 & t \ne 0 \end{cases}$ and $\int_{-\varepsilon}^{\varepsilon} \delta(\tau) d\tau = 1$

Time Shift: $u(t-3); \quad \delta(t+2); \quad u(t+5)$

Square Pulse: $p(t) = u(t+3) - u(t-3)$

Time Scale: $p(2t) = u(2t+3) - u(2t-3) = u(t+\frac{3}{2}) - u(t-\frac{3}{2})$

$p(\frac{t}{2}) = u(\frac{t}{2}+3) - u(\frac{t}{2}-3) = u(t+6) - u(t-6)$

More Typical Signals

Sinusoidal signals: $x_a(t) = A\sin(\omega_0 t) \quad and \quad A\cos(\omega_0 t)$

Complex exponential signals: $x_a(t) = Ae^{j\omega_0 t} = A[\cos(\omega_0 t) + j\sin(\omega_0 t)]$

Phase (radian): $\varphi(t) = \omega_0 t$

Frequency (radian/second): $\omega(t) = \frac{d\varphi(t)}{dt} = \omega_0$

Period (second): $T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$

$$A\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}, \quad A\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Signal with Multiple Sinusoids:

$$x_a(t) = 2\cos(200\pi t - 1) + \sin(120\pi t + \frac{\pi}{3}) - 0.5\cos(80\pi t + 2)$$

Frequency components: $200\pi, 120\pi, 80\pi$

Find the fundamental Frequency and Period: $\omega_0 = MCF\{200\pi, 120\pi, 80\pi\} = 40\pi$

$200\pi = 5\omega_0; \quad 120\pi = 3\omega_0; \quad 80\pi = 2\omega_0$

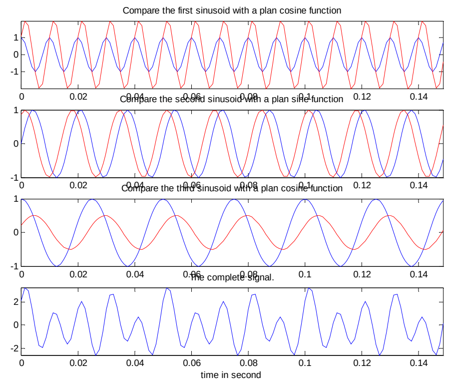
$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{20} \quad (s)$

Phase shift and time shift:

$x_a(t) = \sin(120\pi t + \frac{\pi}{3}); \quad \varphi_0 = \frac{\pi}{3} \quad (rad.); \quad \Delta t = \frac{\varphi_0}{\omega_0} = \frac{1}{360} \quad (s)$

$x_a(t) = \sin(120\pi(t + \frac{\pi}{3 \cdot 120\pi})) = \sin(120\pi(t + \frac{1}{360}))$

Sinusoids: Period, Phase, and Amplitude



Phasor: $Ae^{j\varphi}$

$$2e^{-j}$$

$$e^{j(-\frac{\pi}{2}+\frac{\pi}{3})} = e^{-j\frac{\pi}{6}}$$

$$0.5e^{j(2-\pi)} = 0.5e^{-j1.145}$$

Phasor Addition is only for sinusoids with the same frequency.

Matlab code for plotting Sinusoids:

```
% Multiple Sinusoids:
%
% @ Kefu Xue, Ph.D., June 2006
%
f0=20; % fundamental frequency from Maximum Common Factor: 40pi
T0=1/f0;     % shortest period of the signal
T=T0/40;     % sampling 40 samples over a period of the signal
N=120;       % plot 3 periods
t=[0:N-1]*T; % time sampling vector
s1=2*cos(200*pi*t-1);
s2=sin(120*pi*t+pi/3);
s3=-0.5*cos(80*pi*t+2);
x=s1+s2+s3;
cos200pi=cos(200*pi*t);
sin120pi=sin(120*pi*t);
cos80pi=cos(80*pi*t);
```


Matlab code continue:

```
% compare the signals

figure;subplot(4,1,1);plot(t,cos200pi,'b',t,s1,'r');axis tight;
title('Compare the first sinusoid with a plan cosine function');
subplot(4,1,2);plot(t,sin120pi,'b',t,s2,'r');axis tight;
title('Compare the second sinusoid with a plan sine function');
subplot(4,1,3);plot(t,cos80pi,'b',t,s3,'r');axis tight;
title('Compare the third sinusoid with a plan cosine function');
subplot(4,1,4);plot(t,x,'b');axis tight;
title('The complete signal.');
xlabel('time in second');

% end of file
```


Signal “size”: Energy

Energy of a signal (Note: for any real signal, the energy is always finite):

$$E_x = \int_{-\infty}^{\infty} x_a^2(\tau)d\tau$$

Examples: $x_a(t) = 3 * e^{-2t}u(t)$

$$E_x = \int_{-\infty}^{\infty} 3^2 e^{-2\cdot 2\tau}u(\tau)d\tau = \int_0^{\infty} 9e^{-4\tau}d\tau$$

$$= \frac{9}{-4}e^{-4\tau}\Big|_0^{\infty} = 0 - \frac{9}{-4} = 2.25$$

Numerical approximation: $E_x \cong \sum_{n=0}^{N-1} x_a(nT)\cdot x_a(nT)\cdot T$

Where T is the sampling interval and N is the number of samples which cover majority energy of the signal.

Matlab code for Energy calculation:

```
% Integration approximation and Signal energy and power:
%
% @ Kefu Xue, Ph.D., June 2006
%
% An exponential signal
T=10^-3; % pick a sampling frequency
D=100; % pickup a duration of 100 seconds
N=D/T; % total number of samples
t=[0:N-1]*T;
xe=3*exp(-2*t);
Exe=sum(xe.*xe)*T;
disp(['The approximated total energy is ' num2str(Exe)]);

% end of file
```


Signal "size": Power

For some periodic signals, we can calculate their Power:

$$P_x = \frac{1}{T_0}\int_{T_0} x_a^2(\tau)d\tau$$

Examples:

$$x_a(t) = A\cos(\omega_0 t + \varphi), \quad T_0 = \frac{2\pi}{\omega_0}$$

$$E_x = \frac{1}{T_0}\int_0^{T_0} A^2\cos^2(\omega_0\tau + \varphi)d\tau$$

$$= \frac{A^2}{T_0}\left[\frac{\omega_0\tau + \varphi}{2\omega_0} + \frac{\sin(\omega_0\tau+\varphi)\cos(\omega_0\tau+\varphi)}{2\omega_0}\right]_0^{T_0} = \frac{A^2}{2}$$

$$x_{rms} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} = \frac{\sqrt{2}}{2}A = 0.707\cdot A$$

Where T is the sampling interval and N is the number of samples which cover one period of the signal.

Numerical approximation: $P_x \cong \frac{1}{T_0}\sum_{n=0}^{N-1} x_a(nT)\cdot x_a(nT)\cdot T = \frac{1}{N}\sum_{n=0}^{N-1} x_a^2(nT)$

Matlab code for Power calculation:

```
% Power calculation:
%
f0=20; % fundamental frequency from Maximum Common Factor: 40pi
T0=1/f0; % shortest period of the signal
T=T0/40; % sampling 40 samples over a period of the signal
N=120; % plot 3 periods
t=[0:N-1]*T; % time sampling vector
s1=2*cos(200*pi*t-1);
Ps1=sum(s1(1:40).^2)/40;
display(['The power of a sinusoidal s1 (A^2/2) is equal to ' num2str(Ps1)]);
s2=sin(120*pi*t+pi/3);
Ps2=sum(s2(1:40).^2)/40;
display(['The power of a sinusoidal s2 is equal to ' num2str(Ps2)]);
s3=-0.5*cos(80*pi*t+2);
Ps3=sum(s3(1:40).^2)/40;
display(['The power of a sinusoidal s3 is equal to ' num2str(Ps3)]);
x=s1+s2+s3;
Px=sum(x(1:40).^2)/40;
display(['The power of multiple sinusoids is equal to ' num2str(Px)]);
%
% end of file
```


Signal Comparison and Correlation

To compare the similarity of two signals:

$$\gamma_{xy}(0) = \int_D x_a(\tau)y_a(\tau)d\tau$$

D is the time duration of the signals involved which could be infinite.

Example: Consider the following 3 signals

$$x_a(t) = \cos(\pi t)[u(t) - u(t-2)]$$

$$y_a(t) = \sin(\pi t)[u(t) - u(t-2)]$$

$$z_a(t) = e^{-4t}[u(t) - u(t-3)]$$

$$\gamma_{xx}(0) = \int_0^2 x_a(\tau)x_a(\tau)d\tau = \int_0^2 \cos^2(\pi\tau)d\tau = 1$$

$$\gamma_{xy}(0) = \int_0^2 x_a(\tau)y_a(\tau)d\tau = \int_0^2 \cos(\pi\tau)\sin(\pi\tau)d\tau = 0$$

Signal Comparison and Correlation

$$\gamma_{xz}(0) = \int_0^2 x_a(\tau) z_a(\tau) d\tau = \int_0^2 \cos(\pi\tau) e^{-4\tau} d\tau$$

$$= \frac{e^{-4\tau}}{(-4)^2 + \pi^2}\left(-4\cos(\pi\tau) + \pi\sin(\pi\tau)\right)\Big|_0^2 = 0.1546$$

$$\gamma_{yz}(0) = \int_0^2 y_a(\tau) z_a(\tau) d\tau = \int_0^2 \sin(\pi\tau) e^{-4\tau} d\tau$$

$$= \frac{e^{-4\tau}}{(-4)^2 + \pi^2}\left(-4\sin(\pi\tau) - \pi\cos(\pi\tau)\right)\Big|_0^2 = 0.1214$$

Numerical approximation: $\gamma_{xy}(0) \cong \sum_{n=0}^{N-1} y_a(nT) \cdot x_a(nT) \cdot T$

Signal Comparison and Correlation

$$\gamma_{yy}(0) = \int_0^2 y_a(\tau) y_a(\tau) d\tau = \int_0^2 \sin^2(\pi\tau) d\tau = 1$$

$\gamma_{yy}(0) = \gamma_{xx}(0) = 1$ Unit Energy as a comparison base

$$\gamma_{zz}(0) = \int_0^3 z_a(\tau) z_a(\tau) d\tau = \int_0^3 e^{-8\tau} d\tau = \frac{1}{-8} e^{-8\tau}\Big|_0^3 = 0.124999$$

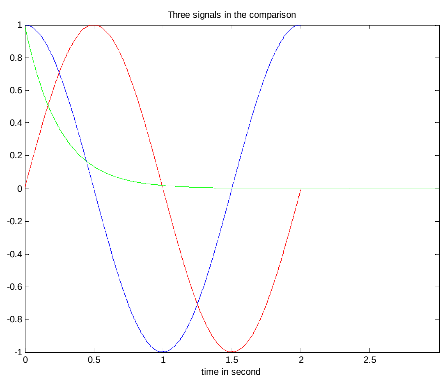
Matlab command> gamaxy=sum(xa.*ya)*T;

Normalized correlation coefficient:

$$\gamma_{xy}(0) = \frac{1}{\sqrt{E_x E_y}} \int_D x_a(\tau) y_a(\tau) d\tau$$

Normalized: $\gamma_{yz}(0) = \frac{0.1214}{\sqrt{0.124999}} = 0.3434;$ $\gamma_{xz}(0) = \frac{0.1546}{\sqrt{0.124999}} = 0.4374$

Signal Comparison



The exponential function is more closely correlated to the cosine function.

Correlation Continue …

Correlation Function:

$$\gamma_{xy}(u) = \int_D x_a(\tau + u) y_a(\tau) d\tau + \int_D x_a(\tau) y_a(\tau - u) d\tau$$

The independent variable u will have a domain to control the number of correlation values to be calculated.

For example:

$$\gamma_{xz}(u) = \int_D x_a(\tau + u) z_a(\tau) d\tau,$$

$$\gamma_{yz}(u) = \int_D y_a(\tau + u) z_a(\tau) d\tau, \quad \text{for } u = 10^{-3} \text{ and } -10^{-3}.$$

Correlation Continue …

$$\gamma_{xz}(10^{-3}) = \int_0^{2-10^{-3}} x_a(\tau+10^{-3}) z_a(\tau) d\tau = \int_{10^{-3}}^{2} x_a(\tau) z_a(\tau-10^{-3}) d\tau$$

$$= \int_{10^{-3}}^{2} \cos(\pi\tau) e^{-4(\tau-10^{-3})} d\tau = e^{4\cdot10^{-3}} \int_{10^{-3}}^{2} \cos(\pi\tau) e^{-4\tau} d\tau$$

$$= \frac{e^{4\cdot10^{-3}} \cdot e^{-4\tau}}{(-4)^2+\pi^2}\left(-4\cos(\pi\tau)+\pi\sin(\pi\tau)\right)\Big|_{10^{-3}}^{2} = 0.1542$$

$$\gamma_{yz}(10^{-3}) = \int_{10^{-3}}^{2} \sin(\pi\tau) e^{-4(\tau-10^{-3})} d\tau$$

$$= \frac{e^{4\cdot10^{-3}} \cdot e^{-4\tau}}{(-4)^2+\pi^2}\left(-4\sin(\pi\tau)-\pi\cos(\pi\tau)\right)\Big|_{10^{-3}}^{2} = 0.1219$$

Matlab function: fgamayz=xcorr(ya,za,1)*T; the results: 0.1209, 0.1214,0.1219 match the integration results.

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Correlation Continue …

$$\gamma_{xz}(-10^{-3}) = \int_{10^{-3}}^{2} x_a(\tau-10^{-3}) z_a(\tau) d\tau = \int_0^{2} x_a(\tau) z_a(\tau+10^{-3}) d\tau$$

$$= \int_0^{2} \cos(\pi\tau) e^{-4(\tau+10^{-3})} d\tau = e^{-4\cdot10^{-3}} \int_0^{2} \cos(\pi\tau) e^{-4\tau} d\tau$$

$$= \frac{e^{-4\cdot10^{-3}} \cdot e^{-4\tau}}{(-4)^2+\pi^2}\left(-4\cos(\pi\tau)+\pi\sin(\pi\tau)\right)\Big|_{0}^{2} = 0.1539$$

$$\gamma_{yz}(-10^{-3}) = \int_0^{2} \sin(\pi\tau) e^{-4(\tau+10^{-3})} d\tau$$

$$= \frac{e^{-4\cdot10^{-3}} \cdot e^{-4\tau}}{(-4)^2+\pi^2}\left(-4\sin(\pi\tau)-\pi\cos(\pi\tau)\right)\Big|_{0}^{2} = 0.1209$$

Matlab function: fgamaxz=xcorr(xa,za,1)*T; the results: 0.1545, 0.1551,0.1547 approximates the integration results.

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Examine the Correlation function:

Consider the sampled signals, y(n), x(n) and z(n):

$$x(n) = x_a(nT), \quad y(n) = y_a(nT), \quad z(n) = z_a(nT)$$

$$\gamma_{xy}(k) = \sum_{k=-L}^{L} x(n+k) y(n), \quad u = kT$$

where k is the index for the time shift in number of sampling intervals. k can go from –L to L to produce a correlation function of 2L+1 samples. Matlab function: gamaxy=xcorr(x,y,L).

- Starting index: Sx, Sy; Ending index: Ex, Ey
- Length: Lx=Ex-Sx+1; Ly=Ey-Sy+1
- Starting: k+Ex=Sy yields Sgama=Sy-Ex
- Ending: k-Sx=Ey yields Egama=Ey+Sx
- Lgama=Lx+Ly-1
- zero-lag (k=0) at: |Sgama|+1 or |Sgama|+2 in Matlab

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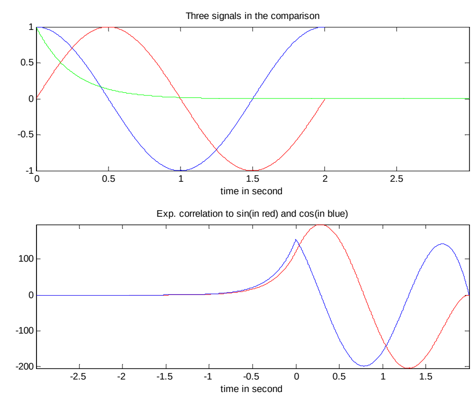
Examine the Correlation function:



The best correlation (peak) between the exponential and sine functions is at time shift t=0.287 s. or a phase shift φ=0.287π. It has better correlation than cosine function with the phase shift.

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