EE322/522
Digital Signal and Systems

1.01: Fourier Series and Line Spectrum

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Fourier Series Representation

- Periodic Signal (Frequency and Period)
- Analysis (Fourier Series Coefficients and Line Spectrum: Phase and Magnitude)
- Synthesis (Sinusoids)

$$x_a(t) = x_a(t+T_0), \quad \omega_0 = \frac{2\pi}{T_0}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x_a(t)e^{-j\omega_0 kt}dt$$

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 kt}$$

Fourier Series of a Real Valued Signal

Real Valued Signal: $x_a(t) = x_a^*(t)$

$$C_k^* = \left(\frac{1}{T_0}\int_{T_0} x_a(t)e^{-j\omega_0 kt}dt\right)^*$$

$$= \frac{1}{T_0}\int_{T_0} x_a^*(t)\left(e^{-j\omega_0 kt}\right)^* dt$$

$$= \frac{1}{T_0}\int_{T_0} x_a(t)e^{j\omega_0 kt}dt = C_{-k}$$

Complex Conjugate Symmetry:

$$C_{-k} = C_k^*$$

$$C_k = |C_k|e^{j\angle C_k}, \quad C_{-k} = |C_k|e^{-j\angle C_k}$$

Synthesis Equation for Real Valued Signal

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 kt}$$

$$= C_0 + \sum_{k=1}^{\infty}\left(C_k e^{j\omega_0 kt} + C_{-k}e^{-j\omega_0 kt}\right)$$

$$= C_0 + \sum_{k=1}^{\infty}\left(|C_k|e^{j\angle C_k}e^{j\omega_0 kt} + |C_k|e^{-j\angle C_k}e^{-j\omega_0 kt}\right)$$

$$= C_0 + \sum_{k=1}^{\infty}\left(|C_n|e^{j(\omega_0 kt+\angle C_k)} + |C_k|e^{-j(\omega_0 kt+\angle C_k)}\right)$$

$$= C_0 + \sum_{k=1}^{\infty} 2|C_k|\cos(\omega_0 kt + \angle C_k)$$

Phasor and Fourier Series Coefficients

Phasor:

$$A\cos(\omega_0 t+\varphi) \;\rightarrow\; \omega_0, Ae^{j\varphi}$$

$$A\sin(\omega_0 t+\varphi) = A\cos(\omega_0 t+\varphi-\frac{\pi}{2}) \;\rightarrow\; \omega_0, Ae^{j(\varphi-\frac{\pi}{2})}$$

Fourier Series Coefficients and Phasor:

$$A\cos(\omega_0 t+\varphi) \;\rightarrow\; 2|C_1|\cos(\omega_0 t+\angle C_1)$$

$$|A| = 2|C_1|; \quad and \quad \varphi = \angle C_1 \quad (\pm\pi, if\;\; A<0)$$

Unit Circle and Principal Phase Value:

$$e^{j\varphi} \quad and \quad -\pi \le \varphi \le \pi$$

Delta function and Sampling

$$x_a(t)\cdot\delta(t) = x_a(0)\cdot\delta(t), \quad x_a(t)\cdot\delta(t-T) = x_a(T)\cdot\delta(t-T)$$

$$s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT),$$

$$x_a(t)\cdot s(t) = x_a(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT) = x_a(nT)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Sample values:

$$\int_{-\varepsilon}^{\varepsilon} x_a(\tau)\cdot\delta(\tau)d\tau = \int_{-\varepsilon}^{\varepsilon} x_a(0)\cdot\delta(\tau)d\tau = x_a(0)\int_{-\varepsilon}^{\varepsilon}\delta(\tau)d\tau = x_a(0)$$

$$\int_{nT-\varepsilon}^{nT+\varepsilon} x_a(\tau)\cdot\delta(\tau-nT)d\tau = \int_{nT-\varepsilon}^{nT+\varepsilon} x_a(nT)\cdot\delta(\tau-nT)d\tau$$

$$= x_a(nT)\int_{nT-\varepsilon}^{nT+\varepsilon}\delta(\tau-nT)d\tau = x_a(nT)$$

Fourier Series of the Sampling Signal

$$s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT), \quad Period = T$$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}$$

$$s(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{T}kt} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j\omega_s kt}$$

$$\omega_s = \frac{2\pi}{T} \quad \text{sampling frequency}$$

Sampling of Sinusoids

$$x_a(t) = 2\cos(200\pi t-1) + \sin(120\pi t+\frac{\pi}{3}) - 0.5\cos(80\pi t+2)$$

$$= 2\cos(200\pi t-1) + \cos(120\pi t+\frac{\pi}{3}-\frac{\pi}{2}) + 0.5\cos(80\pi t+2-\pi)$$

$$= 2\cos(200\pi t-1) + \cos(120\pi t-\frac{\pi}{6}) + 0.5\cos(80\pi t-1.1415)$$

Phasors: $2e^{-j}$, $e^{-j\frac{\pi}{6}}$, $0.5e^{-j1.1415}$

Fourier Series Coefficients: $C_2 = 0.25e^{-j1.1415}, C_{-2} = 0.25e^{j1.1415},$

$$C_3 = \frac{1}{2}e^{-j\frac{\pi}{6}}, C_{-3} = \frac{1}{2}e^{j\frac{\pi}{6}}, C_5 = e^{-j}, C_{-5} = e^{j}$$

Foundamental Frequency(line Spectrum): 40π

Sampling of Sinusoids …

• Sampling frequency must be at least higher than the twice of the highest frequency components in the signal.

$$\omega_s > 2\cdot 5\omega_0 = 400\pi, \quad Let\ \omega_s = 450\pi, then\ T = \frac{2\pi}{\omega_s} = \frac{2}{450}\ s.$$

$$x_a(t)|_{t=nT} = 2\cos(200\pi nT - 1) + \cos(120\pi nT - \frac{\pi}{6}) + 0.5\cos(80\pi nT - 1.1415)$$

$$x(n) = 2\cos(\frac{2\cdot 200\pi}{450}n - 1) + \cos(\frac{2\cdot 120\pi}{450}n - \frac{\pi}{6}) + 0.5\cos(\frac{2\cdot 80\pi}{450}n - 1.1415)$$

$$= 2\cos(\frac{20\cdot 2\pi}{45}n - 1) + \cos(\frac{12\cdot 2\pi}{45}n - \frac{\pi}{6}) + 0.5\cos(\frac{8\cdot 2\pi}{45}n - 1.1415)$$

$$\frac{T_0}{T} = \frac{450}{40} = 11.25\ \text{samples/period(4 periods)}$$

Sampling of Sinusoids …

$$\text{Normalized Frequency}(\theta = \omega T): \quad \frac{l\cdot 2\pi}{N}, \quad \text{(radian)}$$

$$\text{Frequencies}: \quad \omega_5 = 200\pi \Leftrightarrow \theta_5 = \frac{l_5\cdot 2\pi}{N} = \frac{20\cdot 2\pi}{45},$$

$$\omega_3 = 120\pi \Leftrightarrow \theta_3 = \frac{l_3\cdot 2\pi}{N} = \frac{12\cdot 2\pi}{45}, \omega_2 = 80\pi \Leftrightarrow \theta_2 = \frac{l_2\cdot 2\pi}{N} = \frac{8\cdot 2\pi}{45}.$$

The normalized frequencies of a periodic signal must all equal to rational numbers multiplying 2π.

$$\text{Period } N\text{(samples)}: x(n) = x(n \pm mN),$$

$$l_k: \#\ \text{of periods of } k_{th} \text{ frequency in } N \text{ samples.}$$

Sampling of Sinusoids …

• Considering a different sampling frequency

$$\omega_s > 2\cdot 5\omega_0 = 400\pi, \quad \text{Let } \omega_s = 500\pi, \text{ then } T = \frac{2\pi}{\omega_s} = \frac{1}{250}\ s.$$

$$x_a(t)|_{t=nT} = 2\cos(200\pi nT - 1) + \cos(120\pi nT - \frac{\pi}{6}) + 0.5\cos(80\pi nT - 1.1415)$$

$$x(n) = 2\cos(\frac{200\pi}{250}n - 1) + \cos(\frac{120\pi}{250}n - \frac{\pi}{6}) + 0.5\cos(\frac{80\pi}{250}n - 1.1415)$$

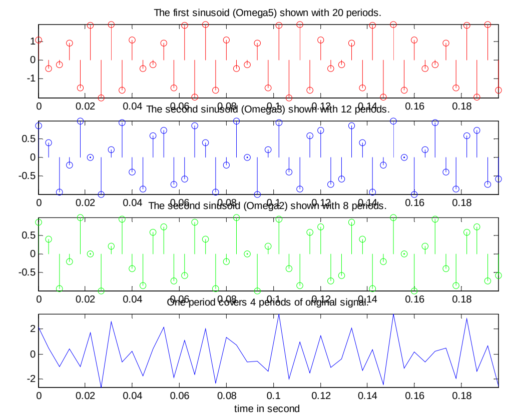
$$= 2\cos(\frac{10\cdot 2\pi}{25}n - 1) + \cos(\frac{6\cdot 2\pi}{25}n - \frac{\pi}{6}) + 0.5\cos(\frac{4\cdot 2\pi}{25}n - 1.1415)$$

$$\text{Normalized Frequencies}: \quad \theta_5 = \frac{10\cdot 2\pi}{25}, \theta_3 = \frac{6\cdot 2\pi}{25}, \theta_2 = \frac{4\cdot 2\pi}{25}.$$

$$\text{Denormalized } (\omega = \frac{\theta}{T} = \theta\cdot f_s = \theta\cdot\frac{\omega_s}{2\pi} = \frac{l}{N}\omega_s): \omega_5 = \frac{10\cdot 500\pi}{25} = 200\pi,$$

$$\omega_3 = \frac{6\cdot 500\pi}{25} = 120\pi, \omega_2 = \frac{4\cdot 500\pi}{25} = 80\pi. \quad (\frac{T_0}{T} = \frac{250}{20} = 12.5\ \text{smpl/prd})$$

Plots (sampling frequency=450π)

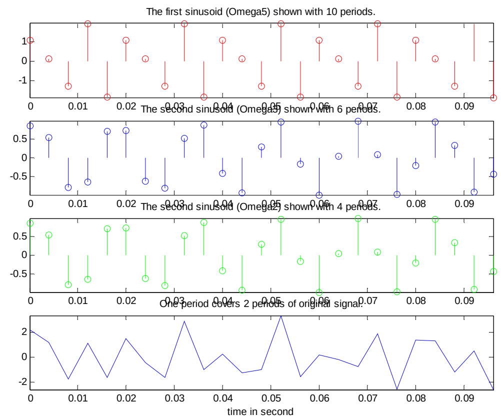


Omega5: 9 (smpls) cover $4\times 2\pi$

Omega3: 15 (smpls) cover $4\times 2\pi$

Omega2: 45 (smpls) cover $8\times 2\pi$

Plots (sampling frequency=500π)



Omega5: 5 (smpls) cover 2x2π

Omega3: 25 (smpls) cover 6x2π

Omega2: 25 (smpls) cover 4x2π

Discrete Fourier Series (DFS):

- Sampling a Periodic Signal (***T*** and ***N***)
- Analysis (DFS)
- Synthesis (Inverse DFS)

$$x(n) = x_a(t)|_{t=nT} = x(n+N), \quad N\cdot T = m\cdot T_0 \quad (m=1, 2, or\ 3, \cdots)$$

$$C_k = \frac{1}{T_0}\int_{T_0} x_a(t)e^{-j\omega_0 kt}dt = \frac{1}{mT_0}\int_{mT_0} x_a(t)e^{-j(\frac{\omega_0}{m})kt}dt$$

$$\approx \frac{1}{NT}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{mT_0}knT}\cdot T = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}X(k)$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} \quad (DFS \text{ and } k\cdot\omega_0 = k\cdot\frac{2\pi}{T_0} = k\cdot\frac{2\pi}{NT}m = k\cdot\frac{\omega_s}{N}m)$$

Matlab function: Xk=fft(xn);

DFS …

- Synthesis (Inverse DFS)

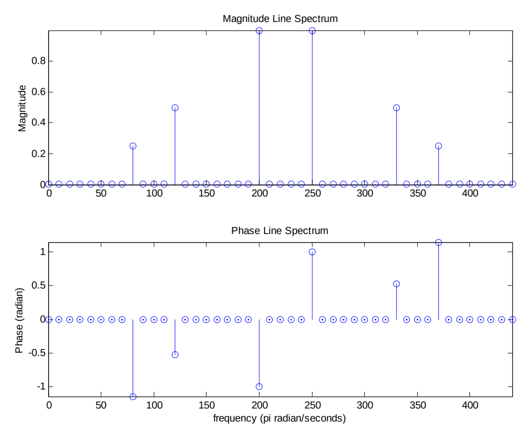
$$x_a(t)|_{t=nT} = x(n) = \sum_{-\infty}^{\infty} C_k e^{j\omega_0 kt}\Big|_{t=nT} = \sum_{-\infty}^{\infty} C_k e^{j\frac{2\pi}{mT_0}knT}$$

$$= \sum_{k=0}^{N-1}\frac{1}{N}X(k)e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi}{N}kn}$$

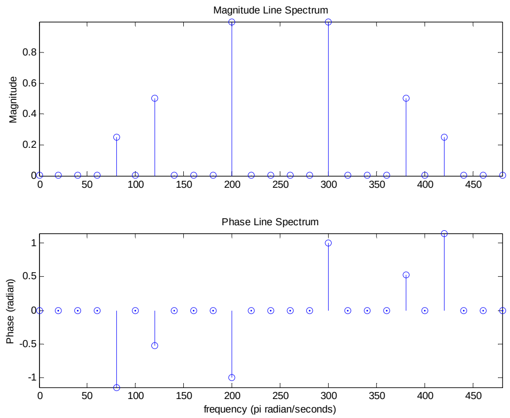
$$x(n) = x(n \pm N) \text{ periods } mT_0$$

Matlab function: xn=ifft(Xk);

DFS of a Periodic Signal (ωs=450π)



DFS of a Periodic Signal (ωs=500π)



DFS Properties:

Periodicity: $x(n \pm N), \quad X(k \pm N)$

Complex Conjugate Symmetry($x(n)$ real valued):

$X(k) = X^*(-k) = X^*(N-k)$, that is,

$|X(k)| = |X(-k)| = |X(N-k)|$

$\angle X(k) = -\angle X(k) = -\angle X(N-k)$

Index:

$k = 0, 1, \cdots N-1.$

$$\theta = k \cdot \frac{2\pi}{N}; \quad \omega_0 = \frac{\omega_s}{N} \cdot m$$

DFS Example: (ωs=450π, *N*=45, m=4)

Peaks (Matlab index): $k = [9\ 13\ 21\ 26\ 34\ 38]$

DFS index $(k-1)$: $[8\ 12\ 20\ 25\ 33\ 37]$

FS index $(k-1)/m$ and $(k-1-N)/m$: $[2\ 3\ 5\ \text{-}5\ \text{-}3\ \text{-}2]$

$X(8) = X^*(-8) = X^*(45-8)$, that is,

$|X(8)| = |X(-8)| = |X(37)|$

$\angle X(8) = -\angle X(8) = -\angle X(37)$

Caculate frequencies:

In FS: $[2\ 3\ 5\ \text{-}5\ \text{-}3\ \text{-}2].*40\pi; \quad \omega_0 = 40\pi$

In DFS: $[8\ 12\ 20\ 20\ 12\ 8].*\frac{450\pi}{45}$; Frequency Resolution: $\frac{\omega_s}{\text{N}}$

Periodic Signal Synthesis

Synthesis Equation:

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 kt} = C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(\omega_0 kt + \angle C_k);$$

Example: Given FS coefficients of a real-valued periodic signal ($T_0 = 10^{-2}$ seconds): $C_0 = 2, C_2 = 1.2e^{-j2}, C_{-4} = 0.6e^{-j}$

$$x_a(t) = 2 + 2 \cdot 1.2\cos(\frac{2\pi}{10^{-2}}2t - 2) + 2 \cdot 0.6\cos(\frac{2\pi}{10^{-2}}4t + 1)$$

$$= 2 + 2.4\cos(400\pi t - 2) + 1.2\cos(800\pi t + 1)$$