EE322/522 Digital Signal and Systems

1.01: Fourier Transform and Spectrum

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Spectrum of a Non-periodic Signal

- FS and Line Spectrum only for periodic signals
- Fourier Transform for periodic and non periodic signals
- Change the expression of a signal in a time-domain (variation of energy) to an expression in the frequency domain
  - Energy Compactness (data compression)
  - Signal Analysis & Synthesis
  - Frequency detection

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Signal Power and Parseval's Theorem

- Recall the Sinusoids' Power Calculation using phasors:

$$x_a(t) = A\cos(\omega_0 t + \varphi); \quad P_x = \frac{1}{T_0}\int_{T_0} A^2\cos^2(\omega_0 t + \varphi)dt = \frac{A^2}{2}$$

$$x_a(t) = 2|C_1|\cos(\omega_0 t + \angle C_1) \Rightarrow C_1 = \frac{A}{2}e^{j\varphi} \text{ and } C_{-1} = \frac{A}{2}e^{-j\varphi}$$

$$P_x = C_1\cdot C_1^* + C_{-1}\cdot C_{-1}^* = |C_1|^2 + |C_{-1}|^2 = 2|C_1|^2 = 2\cdot\left|\frac{A}{2}\right|^2 = \frac{A^2}{2}$$

In General for a Periodic Signal:

$$P_x = \frac{1}{T_0}\int_{T_0} x_a^2(t)dt = \sum_{k=-\infty}^{\infty}|C_k|^2 = |C_0|^2 + 2\cdot\sum_{k=1}^{\infty}|C_k|^2 \quad \text{(for real-valued signal)}$$

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Fourier Transform Definition

$$\text{FS}: C_k = \frac{1}{T_0}\int_{T_0} x_a(t)e^{-j\omega_0 kt}dt, \quad \omega_0 = \frac{2\pi}{T_0}$$

$$\text{IFS}: x_a(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 kt}$$

$$\text{Consider}: \lim T_0 \to \infty \text{ yields } \lim \omega_0 \to 0$$

$$\text{That is}: \omega_0 k \to \omega$$

$$\text{FT}: X_a(\omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt$$

$$\text{IFT}: x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)e^{j\omega t}d\omega$$

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Fourier Transform Examples:

Consider : $\delta(t)$ and $\delta(t-t_0)$

$$\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega 0}dt=1$$

$$\int_{-\infty}^{\infty}\delta(t-t_0)e^{j\omega t}dt=\int_{-\infty}^{\infty}\delta(t-t_0)e^{j\omega t_0}dt=e^{j\omega t_0}\int_{-\infty}^{\infty}\delta(t-t_0)dt=e^{j\omega t_0}$$

Consider : $\text{rect}(t)=u(t+\frac{1}{2})-u(t-\frac{1}{2})$

$$R(\omega)=\int_{-\infty}^{\infty}\text{rect}(t)e^{-j\omega t}dt=\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{-j\omega t}dt=\frac{1}{-j\omega}\int_{-\frac{1}{2}}^{\frac{1}{2}}de^{-j\omega t}=\frac{1}{-j\omega}e^{-j\omega t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$=\frac{e^{-j\omega\frac{1}{2}}-e^{j\omega\frac{1}{2}}}{-j\omega}=\frac{-2j\sin(\frac{\omega}{2})}{-j\omega}=\frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}}=\text{sinc}\left(\frac{\omega}{2}\right)$$

Matlab: sinc(t) is actually sinc(pi*t)

Fourier Transform Examples: (more …)

Consider : $3\cdot\text{rect}(2t)=3\cdot[u(2t+\frac{1}{2})-u(2t-\frac{1}{2})]$

$$=3\cdot[u(2(t+\frac{1}{4}))-u(2(t-\frac{1}{4}))]=3\cdot[u(t+\frac{1}{4})-u(t-\frac{1}{4})]$$

$$R(\omega)=\int_{-\infty}^{\infty}3\cdot\text{rect}(2t)e^{-j\omega t}dt=3\int_{-\frac{1}{4}}^{\frac{1}{4}}e^{-j\omega t}dt=\frac{3}{-j\omega}e^{-j\omega t}\Big|_{-\frac{1}{4}}^{\frac{1}{4}}$$

$$=3\cdot\frac{e^{-j\omega\frac{1}{4}}-e^{j\omega\frac{1}{4}}}{-j\omega}=\frac{3}{2}\cdot\frac{-2j\sin(\frac{\omega}{4})}{\frac{-j\omega}{2}}=\frac{3}{2}\cdot\frac{\sin(\frac{\omega}{4})}{\frac{\omega}{4}}=\frac{3}{2}\cdot\text{sinc}\left(\frac{\omega}{4}\right)$$

Time Shift Properties:

Given : $X_a(\omega)=\int_{-\infty}^{\infty}x_a(t)e^{-j\omega t}dt$ is FT of $x_a(t)$.

Consider : FT of $x_a(t\pm t_0)$

$$\int_{-\infty}^{\infty}x_a(t\pm t_0)e^{-j\omega t}dt\Big|_{\text{let }\tau=t\pm t_0}=\int_{-\infty}^{\infty}x_a(\tau)e^{-j\omega(\tau\mp t_0)}d\tau$$

$$=e^{\pm j\omega t_0}\int_{-\infty}^{\infty}x_a(\tau)e^{-j\omega\tau}d\tau=e^{\pm j\omega t_0}\cdot X_a(\omega)=|X_a(\omega)|e^{j(\angle X_a(\omega)\pm\omega t_0)}$$

$\pm\omega t_0$ (radian) is the phase shift in the FT domain caused by the time shift $\pm t_0$ in the time domain.

Time Scale Properties:

Given : $X_a(\omega)=\int_{-\infty}^{\infty}x_a(t)e^{-j\omega t}dt$ is FT of $x_a(t)$.

Consider : FT of $x_a(\alpha\cdot t)$

$$\int_{-\infty}^{\infty}x_a(\alpha\cdot t)e^{-j\omega t}dt\Big|_{\text{let }\tau=\alpha t}=\frac{1}{\alpha}\int_{-\infty}^{\infty}x_a(\tau)e^{-j\omega(\frac{\tau}{\alpha})}d\tau$$

$$=\frac{1}{|\alpha|}\int_{-\infty}^{\infty}x_a(\tau)e^{-j\frac{\omega}{\alpha}\tau}d\tau=\frac{1}{|\alpha|}X_a(\frac{\omega}{\alpha})$$

Example (time - frequency scale) :

$$x_a(t)=5\cdot\text{rect}(\frac{t}{4}-0.5)=5\cdot\text{rect}(\frac{t-2}{4})$$

$$\leftrightarrow X_a(\omega)=4\cdot5\cdot\text{sinc}\left(\frac{4\omega}{2}\right)e^{-j2\omega}=20\cdot\text{sinc}(2\omega)e^{-j2\omega}$$

Frequency Shift Properties:

Given: $X_a(\omega)=\int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt$ is FT of $x_a(t)$.

Consider FT: $X_a(\omega \pm \omega_0)$

$$X_a(\omega\pm\omega_0)=\int_{-\infty}^{\infty} x_a(t)e^{-j(\omega\pm\omega_0)t}dt=\int_{-\infty}^{\infty}[x_a(t)e^{\mp j\omega_0 t}]e^{-j\omega t}dt$$

$$X_a(\omega\pm\omega_0)\leftrightarrow x_a(t)e^{\mp j\omega_0 t}=x_a(t)\cdot\cos(\omega_0 t)\mp jx_a(t)\cdot\sin(\omega_0 t)$$

The frequency shift in the FT domain produces a multiplication of complex exponential in the time domain.

AM Radio (Example):

$$X_a(\omega+\omega_0)+X_a(\omega-\omega_0)\leftrightarrow x_a(t)\cdot 2\cos(\omega_0 t)$$

Speach (modulating) signal, $x_a(t)$ and Carrier Signal, $\cos(\omega_0 t)$

Duality:

Given: $X_a(\omega)=\int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt$ is FT of $x_a(t)$.

Then: $X_a(t)\leftrightarrow 2\pi\cdot x_a(-\omega)$

$$X_a(t)=_{t=\omega;u=t}=\int_{-\infty}^{\infty} x_a(u)e^{-jtu}du=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\cdot x_a(u)e^{-jtu}du$$

$$=_{\omega=-u}\frac{1}{2\pi}\int_{\infty}^{-\infty}2\pi\cdot x_a(-\omega)e^{jt\omega}d(-\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\cdot x_a(-\omega)e^{jt\omega}d\omega$$

Example:

$$\mathrm{rect}(t)\leftrightarrow \mathrm{sinc}\left(\frac{\omega}{2}\right);\quad \mathrm{sinc}\left(\frac{t}{2}\right)\leftrightarrow 2\pi\cdot\mathrm{rect}(-\omega)=2\pi\cdot\mathrm{rect}(\omega);$$

$$\frac{1}{2\pi}\mathrm{sinc}\left(\frac{t}{2}\right)\leftrightarrow \mathrm{rect}(\omega);\ \mathrm{sinc}\left(2\cdot\frac{t}{2}\right)=\mathrm{sinc}(t)\leftrightarrow \pi\cdot\mathrm{rect}(\frac{\omega}{2})$$

Fourier Transform of Sampling Signal:

$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j\omega_s kt},\quad \omega_s=\frac{2\pi}{T}$$

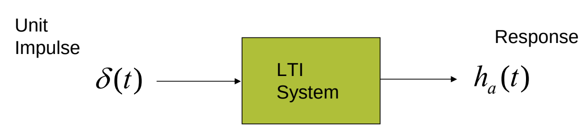
$$S(\omega)=\int_{-\infty}^{\infty}\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j\omega_s kt}e^{-j\omega t}dt=\frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j\omega_s kt}e^{-j\omega t}dt$$

$$=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_s k)=\omega_s\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_s k)$$

Consider: $\delta(t)\leftrightarrow 1;\quad 1\leftrightarrow 2\pi\cdot\delta(-\omega)=2\pi\cdot\delta(\omega);$

$1\cdot e^{j\omega_0 t}\leftrightarrow 2\pi\cdot\delta(\omega-\omega_0)$ (Duality & frequency shift)

Convolution: (Linear Time Invariant Systems and Unit Impulse Response)



For a LTI system, $h_a(t)$ completely describes the Dynamic behavior of the system.

Linear: $a\delta(t)\rightarrow ah_a(t)$; Time Invariant: $\delta(t-t_0)\rightarrow h_a(t-t_0)$

LTI: $a\delta(t)+b\delta(t-t_0)\rightarrow ah_a(t)+bh_a(t-t_0)$

Any Input Signal: $x_a(t)=\lim_{T\to 0}\sum_{n=-\infty}^{\infty}x_a(nT)\delta(t-nT)=\int_{-\infty}^{\infty}x_a(\tau)\delta(t-\tau)d\tau$

Convolution and LTI Systems:

LTI Response: $y_a(t) = \lim_{T\to 0} \sum_{n=-\infty}^{\infty} x_a(nT)h_a(t-nT) = \int_{-\infty}^{\infty} x_a(\tau)h_a(t-\tau)d\tau$

$= \int_{-\infty}^{\infty} x_a(t-\tau)h_a(\tau)d\tau = x_a(t) * h_a(t)$ (Linear Convolution Integral)

Evaluations:

$y_a(-5) = \int_{-\infty}^{\infty} x_a(\tau)h_a(-5-\tau)d\tau = \int_{-\infty}^{\infty} x_a(-5-\tau)h_a(\tau)d\tau$

$y_a(5) = \int_{-\infty}^{\infty} x_a(\tau)h_a(5-\tau)d\tau = \int_{-\infty}^{\infty} x_a(5-\tau)h_a(\tau)d\tau$

$y_a(0) = \int_{-\infty}^{\infty} x_a(\tau)h_a(-\tau)d\tau = \int_{-\infty}^{\infty} x_a(-\tau)h_a(\tau)d\tau$

Convolution Example:

Example: $x_a(t) = e^{-2t} \cdot u(t-1)$ and $h_a(t) = 2 \cdot rect(\frac{t-3}{2})$

Note: $S_x = 1;\ E_x = \infty;\ S_h = 2;\ E_h = 4.\ (S_y = S_x + S_h;\ E_y = E_x + E_h;)$

$y_a(t) = x_a(t) * h_a(t) = \int_{-\infty}^{\infty} x_a(\tau)h_a(t-\tau)d\tau$

$= \int_{-\infty}^{\infty} e^{-2\tau} \cdot u(\tau-1) \cdot 2 \cdot rect(\frac{t-\tau-3}{2})d\tau$

$$= \begin{cases} 0, \quad t < S_x + S_h = 3 \\ \int_{S_x=1}^{t-S_h=t-2} 2e^{-2\tau}d\tau = -e^{-2\tau}\Big|_1^{t-2} = e^{-2} - e^{4-2t}, \quad 3 \le t < 5 \\ \int_{t-E_h=t-4}^{t-S_h=t-2} 2e^{-2\tau}d\tau = -e^{-2\tau}\Big|_{t-4}^{t-2} = (e^8 - e^4)e^{-2t}, \quad 5 \le t \end{cases}$$

Convolution and Correlation:

Convolution: $y_a(t) = x_a(t) * h_a(t) = \int_{-\infty}^{\infty} x_a(\tau)h_a(t-\tau)d\tau$

Correlation: $\gamma_{xh}(u) = \int_{-\infty}^{\infty} x_a(t+u)h_a(u)du$

Considering: $u = -\tau$

$\gamma_{xh}(-\tau) = \int_{\infty}^{-\infty} x_a(t-\tau)h_a(-\tau)d(-\tau)$

$= \int_{-\infty}^{\infty} x_a(t-\tau)h_a(-\tau)d\tau = x_a(t) * h_a(-t)$

Matlab: xcorr(x, y) = fliplr(conv(x, fliplr(y)));

Fourier Transform of Convolution:

$y_a(t) = x_a(t) * h_a(t) = \int_{-\infty}^{\infty} x_a(\tau)h_a(t-\tau)d\tau$

$Y_a(\omega) = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} x_a(\tau)h_a(t-\tau)d\tau)e^{-j\omega t}dt$

$= \int_{-\infty}^{\infty} x_a(\tau)(\int_{-\infty}^{\infty} h_a(t-\tau)e^{-j\omega t}dt)d\tau$

$= \int_{-\infty}^{\infty} x_a(\tau)H_a(\omega)e^{-j\omega\tau}d\tau = H_a(\omega)\int_{-\infty}^{\infty} x_a(\tau)e^{-j\omega\tau}d\tau$

$= H_a(\omega) \cdot X_a(\omega)$

Duality: $x_a(t) \cdot h_a(t) \leftrightarrow \frac{1}{2\pi} H_a(\omega) * X_a(\omega)$

Fourier Transform of Correlation Function:

$$\gamma_{xy}(t) = \int_{-\infty}^{\infty} x_a(\tau) y_a(\tau + t) d\tau$$

$$\Gamma_{xy}(\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_a(\tau) y_a(\tau + t) d\tau \right) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x_a(\tau) \left(\int_{-\infty}^{\infty} y_a(\tau + t) e^{-j\omega t} dt \right) d\tau$$

$$= \int_{-\infty}^{\infty} x_a(\tau) Y_a(\omega) e^{j\omega\tau} d\tau = Y_a(\omega) \int_{-\infty}^{\infty} x_a(\tau) e^{j\omega\tau} d\tau$$

$$= Y_a(\omega) \cdot X_a(-\omega) =_{x_a(t)\ \text{real-valued}} Y_a(\omega) \cdot X_a^*(\omega)$$

$$\text{Power Spectrum}: \gamma_{xx}(t) \leftrightarrow X_a(\omega) \cdot X_a^*(\omega) = |X_a(\omega)|^2$$

Correlation Function Example:

$$x_a(t) = 2 \cdot rect(\frac{t}{2} - 1), \quad y_a(t) = e^{-2t} \cdot u(t)$$

$$x_a(t) = 2 \cdot rect(\frac{t-2}{2}) \leftrightarrow X_a(\omega) = 2 \cdot \frac{1}{(1/2)} \cdot \text{sinc}(\frac{\omega}{(1/2) \cdot 2}) \cdot e^{-j2\omega}$$

$$\text{Delay}: \tau_0 = -2; \text{Scale}: \alpha = \frac{1}{2}. \Rightarrow \quad X_a(\omega) = 4 \cdot \text{sinc}(\omega) \cdot e^{-j2\omega}$$

$$y_a(t) = e^{-2t} \cdot u(t) \leftrightarrow Y_a(\omega) = \frac{1}{2 + j\omega} \quad \text{using FT definition}$$

$$\Gamma_{xy}(\omega) = Y_a(\omega) \cdot X_a^*(\omega) = \frac{4 \cdot \text{sinc}(\omega)}{2 + j\omega} \cdot e^{j2\omega} = \frac{4 \cdot \text{sinc}(\omega)}{\sqrt{2^2 + \omega^2} \cdot e^{j\frac{\omega}{2}}} \cdot e^{j2\omega}$$

$$= \frac{4 \cdot \text{sinc}(\omega)}{\sqrt{2^2 + \omega^2}} \cdot e^{j(2\omega - \frac{\omega}{2})} = \frac{4 \cdot \text{sinc}(\omega)}{\sqrt{2^2 + \omega^2}} \cdot e^{j(\frac{3\omega}{2})}$$

Example (continuing): …

Power Spectrum:

$$\Gamma_{xx}(\omega) = X_a(\omega) \cdot X_a^*(\omega) = |X_a(\omega)|^2 = 16 \cdot \text{sinc}^2(\omega)$$

$$\Gamma_{yy}(\omega) = Y_a(\omega) \cdot Y_a^*(\omega) = \frac{1}{2 + j\omega} \cdot \frac{1}{2 - j\omega} = \frac{1}{2^2 + \omega^2}$$

$$|\Gamma_{xy}(\omega)|^2 = \Gamma_{xy}(\omega) \cdot \Gamma_{xy}^*(\omega) = Y_a(\omega) \cdot X_a^*(\omega) \cdot Y_a^*(\omega) \cdot X_a(\omega)$$

$$= |X_a(\omega)|^2 \cdot |Y_a(\omega)|^2 = \frac{16 \cdot \text{sinc}^2(\omega)}{2^2 + \omega^2} = \Gamma_{xx}(\omega) \cdot \Gamma_{yy}(\omega)$$

Observation:

DC Power:

$$\Gamma_{xx}(0) = X_a(0) \cdot X_a^*(0) = |X_a(0)|^2 = 16 \cdot \text{sinc}^2(0) = 16$$

$$\Gamma_{xx}(0) = \int_{-\infty}^{\infty} \gamma_{xx}(t) e^{-j0t} dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_a(\tau) x_a(\tau + t) d\tau \right] dt$$

$$= \int_{-\infty}^{\infty} x_a(\tau) \left[\int_{-\infty}^{\infty} x_a(\tau + t) dt \right] d\tau = \int_{-\infty}^{\infty} x_a(\tau) D_x d\tau = D_x^2$$

$$D_x = \int_{-\infty}^{\infty} x_a(t) dt = \int_{-\infty}^{\infty} 2 \cdot \text{rect}(\frac{t-2}{2}) dt = \int_1^3 2 \cdot dt = 2t \,|_1^3 = 4$$

$$\Gamma_{yy}(0) = Y_a(0) \cdot Y_a^*(0) = \frac{1}{2^2 + 0^2} = \frac{1}{4}$$

$$|\Gamma_{xy}(0)|^2 = |X_a(0)|^2 \cdot |Y_a(0)|^2 = \frac{16 \cdot \text{sinc}^2(0)}{2^2 + 0^2} = \frac{16}{4} = 4$$