EE322/522
Digital Signal and Systems

1.01: Discrete time Fourier Transform and DFT

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DtFT Discrete-time Fourier Transform

- Fourier Transform of the Sampling Signal
- Fourier Transform of a sampled Signal
- Periodicity of DtFT
- Aliasing errors revisited
- Sampling Theorem: Nyquist rate, Bandwidth and Folding frequency
- Normalized frequency
- Anti-aliasing filter

Fourier Transform of Discrete-time Signal:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\omega_s kt}, \quad \omega_s = \frac{2\pi}{T}$$

$$S(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega_s kt} e^{-j\omega t} dt = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_s k) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_s k)$$

Consider : $x_s(t) = x_a(t)|_{t=nT} = x(n) = x_a(t) \cdot s(t)$

DtFT : $X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * S(\omega) = \frac{\omega_s}{2\pi} X_a(\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_s k)$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - \omega_s k)$$

Fourier Transform of Discrete-time Signal:

DtFT Calculation :

$$X_s(\omega) = \int_{-\infty}^{\infty} x_a(t) \cdot s(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} x_a(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x_a(nT) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \cdot e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} x_a(nT) \int_{-\infty}^{\infty} \delta(t - nT) \cdot e^{-j\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT) \cdot e^{-j\omega nT}$$

DtFT Expression : $X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\theta \cdot n}$

Normalized frequency and the Unit Circle : $\theta = \omega \cdot T$ and $e^{-j\theta}$

$$X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * S(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - \omega_s k)$$

$$\Rightarrow X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\theta \cdot n}$$

Inverse DtFT Calculation:

DtFT Expression : $X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\theta \cdot n}$

$X(e^{j\theta})$ is periodic with period equal to $T_0 = 2\pi$.

IDtFT : $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta \cdot n} d\theta$

Duality? Consider the fundamental frequency of $X(e^{j\theta})$ be $(\frac{2\pi}{T_0} = 1)$ and $x(n)$ can be considered Fourier Series Coefficients.

Sampling in the time domain ←→ Linear mapping in the frequency domain

$t = nT \quad \Leftrightarrow \quad \theta = \omega \cdot T$

Spectrum Periodicity :

$$X(e^{j(\theta \pm k \cdot 2\pi)}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j(\theta \pm k \cdot 2\pi) \cdot n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\theta \cdot n} e^{j \mp k \cdot 2\pi \cdot n} = X(e^{j\theta})$$

$$X_s(\omega \pm \omega_s m) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega \pm \omega_s m - \omega_s k)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - \omega_s (k \mp m)) = X_s(\omega)$$

Bandwidth and Nyquist frequency of a real signal:

$f_{B-3dB} \Rightarrow \quad 10 \cdot \log_{10} |X(2\pi \cdot f_{B-3dB})|^2 = 10 \cdot \log_{10} \left(\frac{|X(\omega)|^2_{\max}}{2} \right)$ and

$$10 \cdot \log_{10} |X(\omega)|^2 < 10 \cdot \log_{10} \left(\frac{|X(\omega)|^2_{\max}}{2} \right)$$

for any frequencies $\omega > 2\pi \cdot f_{B-3dB}$, in other words :

$$10 \cdot \log_{10} |X(2\pi \cdot f_{B-3dB})|^2 = 10 \cdot \log_{10} \left(|X(\omega)|^2_{\max} \right) - 10 \cdot \log_{10}(2)$$

$$10 \cdot \log_{10} |X(\omega_{B-3dB})|^2 - 10 \cdot \log_{10} \left(|X(\omega)|^2_{\max} \right) = -10 \cdot \log_{10}(2) \quad (dB)$$

$$10 \cdot \log_{10} \left(\frac{|X(\omega_{B-3dB})|^2}{|X(\omega)|^2_{\max}} \right) = 20 \cdot \log_{10} \left(\frac{|X(\omega_{B-3dB})|}{|X(\omega)|_{\max}} \right) = -3 \quad (dB)$$

Nyquist frequency of the signal : $f_N = 2 \cdot f_{B-3dB}$

Folding frequency of a Digital System

A Digital System with a sampling interval :

T and $f_s = \frac{1}{T}$ (Hz)

Folding Frequency is the highest frequency component represented in the Digital System without aliasing error :

$f_{\text{fold}} = \frac{f_s}{2}$ (Hz) or $\omega_{\text{fold}} = \frac{\omega_s}{2}$ (radian/second)

Normalized folding frequency $(\theta = \omega \cdot T)$:

$$\theta = \frac{\omega_s}{2} \cdot T = \frac{2\pi}{2T} \cdot T = \pi$$

Sampling Theorem:

$$f_s > f_N = 2 \cdot f_B$$

or

$$f_{\text{fold}} > f_B \quad \text{(Hz)}$$

or

$$\omega_{\text{fold}} > \omega_B \quad \text{(radian/second)}$$

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Aliasing error (Frequency Ambiguity):

If $f_s \le f_N = 2 \cdot f_B$, then the normalized frequency

$\theta_B = 2\pi \cdot f_B \cdot T = \dfrac{\pi \cdot 2 \cdot f_B}{f_s} \ge \pi$, on bottom of the Unit Circle.

The frequency is folded as $2\pi - \theta_B$ on top of the Unit Circle.

$$\omega_B' = (2\pi - \theta_B) \cdot \frac{1}{T} = \frac{2\pi}{T} - \omega_B \quad \text{(radian/second)}$$

or

$f_B' = f_s - f_B$ (Hz) (folded according to f_B)

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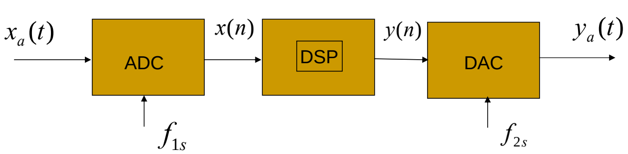
Aliasing Error (Examples):



In a Digital Signal Processing (DSP) System:

f_{1s} don't have to be equal to f_{2s} (multi-rate DSP)

$x_a(t) = 3\cos(200\pi t) + 2\sin(600\pi t - 1)$

sampled at $f_s = 400$ (Hz)

$f_N = ?$ and $f_{\text{fold}} = ?$ and Aliasing error?

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Aliasing error (Example continue):

$$f_N = 2 \cdot f_B = 2 \cdot 300 = 600 \text{ (Hz)}$$

$$f_{\text{fold}} = \frac{f_s}{2} = \frac{400}{2} = 200 \text{ (Hz)}$$

Yes, there is aliasing error in the sampling process.

Reason: $f_s \le f_N$ or $f_{\text{fold}} \le f_B$

$$x(n) = x_a(t)\big|_{t=\frac{n}{400}} = 3\cos(\frac{200\pi}{400}n) + 2\sin(\frac{600\pi}{400}n - 1)$$

$$= 3\cos(\frac{2\pi}{4}n) + 2\cos(\frac{6\pi}{4}n - (1 + \frac{\pi}{2}))$$

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Aliasing error (Example continue):

$$= 3\cos(\frac{2\pi}{4}n) + 2\cos(\frac{6\pi}{4}n - (1+\frac{\pi}{2}))$$

$$= 3\cos(\frac{2\pi}{4}n) + 2\cos((\frac{6\pi}{4} - 2\pi)n - (1+\frac{\pi}{2}))$$

$$= 3\cos(\frac{2\pi}{4}n) + 2\cos(\frac{-2\pi}{4}n - (1+\frac{\pi}{2}))$$

$$= 3\cos(\frac{2\pi}{4}n) + 2\cos(\frac{2\pi}{4}n + (1+\frac{\pi}{2}))$$

$$= 1.7\cos(\frac{2\pi}{4}n + 0.6871)$$

Aliasing error (Example continue):

Reconstructed signal with

the same sampling frequency, $f_s = 400$ (Hz)

$$y_a(t) = 1.7\cos(\frac{2\pi}{4} \cdot f_s n + 0.6871)$$

$$= 1.7\cos(\frac{2\pi}{4} \cdot 400n + 0.6871)$$

$$= 1.7\cos(200\pi n + 0.6871)$$

Aliasing error (Another Example):

Consider an analog signal with following frequency components:

$\omega_1 = 300\pi$, $\omega_2 = 1170\pi$, $\omega_3 = 720\pi$, $\omega_4 = 450\pi$

and their corresponding phasors are

$2e^{-j\frac{\pi}{3}}$, $e^{-j\frac{\pi}{2}}$, $0.5e^{-j\frac{\pi}{4}}$, $e^{j\frac{\pi}{2}}$

If the signal is sampled/reconstructed at $f_s = 360$ (Hz), find the phasors of the reconstructed signal.

Aliasing error (Example continue):

Normalizd requency components:

$$\theta_1 = \omega_1 T = \frac{300\pi}{360} = \frac{30 \cdot 2\pi}{72},$$

$$\theta_2 = \omega_2 T = \frac{1170\pi}{360} = \frac{117 \cdot 2\pi}{72} > \pi,$$

$$\theta_3 = \omega_3 T = \frac{720\pi}{360} = \frac{72 \cdot 2\pi}{72} > \pi,$$

$$\theta_4 = \omega_4 T = \frac{450\pi}{360} = \frac{45 \cdot 2\pi}{72} > \pi.$$

Aliasing error (folding):

Folded ormalizd requency components:

$$\theta_1 = \frac{30\cdot 2\pi}{72},$$

$$\theta_2' = \frac{117\cdot 2\pi}{72} - 2\cdot 2\pi = \frac{-27\cdot 2\pi}{72},$$

$$\theta_3' = \frac{72\cdot 2\pi}{72} - 2\pi = 0,$$

$$\theta_4' = \frac{45\cdot 2\pi}{72} - 2\pi = \frac{-27\cdot 2\pi}{72}.$$

Aliasing error (continue):

θ_2' and θ_4' appears as the same frequency component:

$(e^{-j\frac{\pi}{2}})^* + (e^{j\frac{\pi}{2}})^* = 0$ is the combined phasor.

θ_3' appears as the DC component, the reconstructed signal becomes:

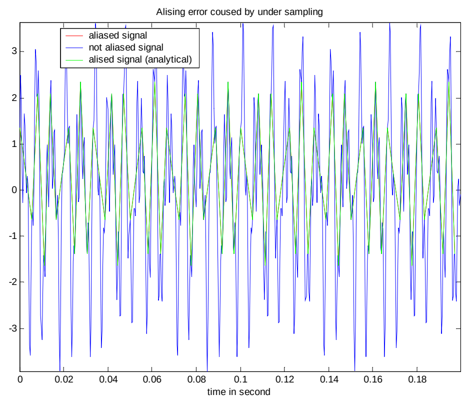
$$y_a(t) = 0.5\cos(0\cdot f_s\cdot t - \frac{\pi}{4}) + 2\cos(\frac{30\cdot 2\pi}{72}\cdot f_s\cdot t - \frac{\pi}{3})$$

$$= 0.5\cos(\frac{\pi}{4}) + 2\cos(300\pi t - \frac{\pi}{3})$$

Compared with the original signal:

$$x_a(t) = 2\cos(300\pi t - \frac{\pi}{3}) + \sin(1170\pi t) + 0.5\cos(720\pi t - \frac{\pi}{4}) - \sin(450\pi t)$$

Aliasing error (continue):



A signal sampled at different frequencies, blue sampled at 2880 Hz and the red (green) sampled at 360 Hz. The green is the analytical result which matches the data from the directly sampled signal (red).

Discrete Fourier Transform (DFT):

DtFT: $X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\theta\cdot n}$

Frequency sampling (N samples on the Unit Circle):

$$\theta = \frac{2\pi}{N}k, \quad k = 0,1,\cdots,N-1.$$

Data Truncation (period):

$$x_p(n) = x_p(n \pm N), \quad n = 0,1,\cdots,N-1.$$

$$\text{DFT}: X(e^{j\frac{2\pi}{N}k}) = X(k) = \sum_{n=0}^{N-1} x_p(n)\cdot e^{-j\frac{2\pi}{N}k\cdot n}$$

$$\text{IDFT}: x_p(n) = \sum_{k=0}^{N-1} X(k)\cdot e^{j\frac{2\pi}{N}k\cdot n}, \quad k = 0,1,\cdots,N-1.$$

DFT (error caused by truncation):

$$\text{Data Truncation}: \; D = NT \text{ and } \tau = \frac{D}{2}$$

$$x_p(n) = x_s(t)\cdot \text{rect}(\frac{t-\tau}{D})$$

$$FT\left\{x_s(t)\cdot \text{rect}(\frac{t-\tau}{D})\right\} = \frac{1}{2\pi} FT\{x_s(t)\} * FT\left\{\text{rect}(\frac{t-\tau}{D})\right\}$$

$$= \frac{1}{2\pi\cdot T}\sum_{k=-\infty}^{\infty} X_a(\omega-\omega_s k) * D\cdot \text{sinc}(\frac{\omega D}{2})\cdot e^{-j\tau\omega}$$

$$= \frac{N}{2\pi}\sum_{k=-\infty}^{\infty} X_a(\omega-\omega_s k) * \text{sinc}(\frac{\omega D}{2})\cdot e^{-j\tau\omega}$$

$\left|\text{sinc}(\frac{\omega D}{2})\right|$ causes ripple distortion (Leakage error) in $X_s(\omega)$

Note: $\lim_{D\to\infty} \text{sinc}(\frac{\omega D}{2}) = \delta(\omega)$

DFT (error caused by frequency sampling):

$$\theta = \frac{2\pi}{N}k, \quad k = 0,1,\cdots,N-1. \text{ and } \theta = \omega\cdot T$$

$$\omega = \frac{\theta}{T} = \frac{2\pi}{NT}k = \frac{\omega_s}{N}k = \frac{2\pi}{D}k,$$

Finite frequency resolution:

$$\frac{\omega_s}{N} \text{ (radian/sec.)}, \; \frac{f_s}{N} = \frac{1}{D} \text{ (Hz)}, \; \frac{2\pi}{N} \text{ (radian)}$$

DFT (Time domain periodic):

$$\tilde{X}_s(\omega)\Big|_{\frac{\omega_s}{N}k} = \frac{N}{2\pi}\sum_{m=-\infty}^{\infty} \tilde{X}_a(\frac{\omega_s}{N}k-\omega_s m)$$

Consider frequency sampling: $\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{\omega_s}{N}k)$

$$\sum_{m=-\infty}^{\infty}\delta(t-ma) \leftrightarrow \omega_a\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_a k), \text{ with } \quad \omega_a = \frac{2\pi}{a}$$

Let $\omega_a = \frac{\omega_s}{N}$, then $a = \frac{2\pi\cdot N}{\omega_s} = N\cdot T = D$

$$\frac{D}{2\pi}\sum_{m=-\infty}^{\infty}\delta(t-mD) \leftrightarrow \sum_{k=-\infty}^{\infty}\delta(\omega-\frac{\omega_s}{N}k)$$

$$x_p(t) = x_s^T(t) * \frac{D}{2\pi}\sum_{m=-\infty}^{\infty}\delta(t-mD) = \frac{N\cdot T}{2\pi}\sum_{n=-\infty}^{\infty} x_s^T(t-mD)$$

DFT Interpretation:

$$x_p(n) = \sum_{n=-\infty}^{\infty} x_s(t-mD)$$

$$X(k) = \frac{1}{T}\sum_{m=-\infty}^{\infty}\tilde{X}_a(\frac{\omega_s}{N}k-\omega_s m)$$

DFT is performed over one period of N samples of a signal which is labeled with n=0,1, …., N-1.

DFT are samples of the original spectrum with possibly aliasing errors and ripple errors. An scale factor of 1/T is introduced.

DFT Example:

$$x_a(t) = 500e^{-1000|t|} \leftrightarrow X_a(\omega) = \frac{10^6}{10^6 + \omega^2}$$

Select sampling frequency such that the aliasing error amplitude at the folding prequency is less than -80 dB

$$|X_a(\omega)|_{\max} = |X_a(0)| = 1$$

$$20\log_{10}\left(\frac{\left|X_a(\frac{\omega_s}{2})\right|}{|X_a(0)|}\right) < -80 \text{ dB}$$

$$20\log_{10}\left(\frac{10^6}{10^6 + \left[\frac{\omega_2}{2}\right]^2}\right) < -80 \text{ dB}$$

DFT Example (continue):

$$20\log_{10}(10^6) - 20\log_{10}\left(10^6 + \left[\frac{\omega_s}{2}\right]^2\right) < -80 \text{ dB}$$

$$-20\log_{10}\left(10^6 + \left[\frac{\omega_s}{2}\right]^2\right) < -80 - 120 \text{ dB}$$

$$\log_{10}\left(10^6 + \left[\frac{\omega_s}{2}\right]^2\right) > 10 \Rightarrow 10^6 + \left[\frac{\omega_s}{2}\right]^2 > 10^{10}$$

$$\omega_s > 2\sqrt{10^{10} - 10^6} = 31829 \cdot 2\pi \text{ (Nyquist rate)}$$

$$\Rightarrow f_s = 32 \text{ KHz}$$

DFT Example (continue):

Desired frequency resolution : $f_r = \frac{f_s}{N} = \frac{32000}{N} = 200$ Hz

$N = 160$ and $D = N \cdot T = \frac{1}{f_r}$ (second) $= 5$ (ms)

Take the samples from $-\frac{D}{2}$ to $\frac{D}{2}$ including maximum signal energy.

Plot the magnitude spectrum of both FT $X_a(k \cdot 200)$ and $T \cdot X(k \cdot 200)$ in one figure.

DFT Example (continue): Trunacation

Energy Consideration :

$$E_x = \int_{-\infty}^{\infty} x_a^2(t)dt = \int_{-\infty}^{\infty} 500^2 e^{-2000|t|}dt$$

$$= 2\int_0^{\infty} 500^2 e^{-2000t}dt = 250 \text{ (square unit)}$$

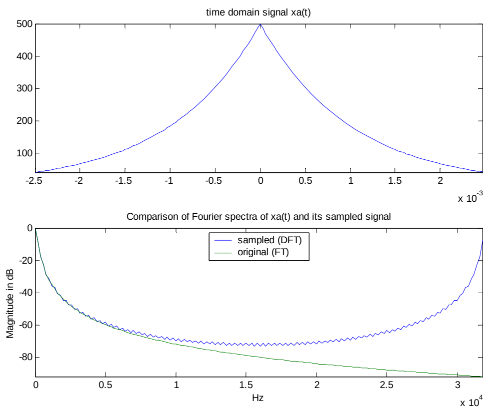
What is the percent of energy for the 5 (ms) duration?

$$E_x = 2\int_0^{2.5\times10^{-3}} 500^2 e^{-2000t}dt = \frac{500000}{-2000} e^{-2000t}\Big|_0^{2.5\times10^{-3}}$$

$$= -250(e^{-5} - 1) = 0.9933 \cdot 250 \text{ (square unit)}$$

The 5 ms duration covers 99.33% of signal energy!

DFT Example (continue):



The aliasing error at the folding frequency, 16 KHz is less than -80 dB.

DFT (Window Functions):

Data Duration: $D = NT$ covers the signal duration with maximum energy.

$x_p(n) = x(n) \cdot w(n), \quad n = 0, 1, \cdots, N-1.$

$$\frac{1}{2\pi} X(e^{j\theta}) * W(e^{j\theta})$$

It is desirable $W(e^{j\theta})$ has less ripples energy (lower ripple amplitude) and narrower main lobe (better frequency resolvability).

Window Functions:

Rectangular window: $w_{rect}(n) = 1, \quad n = 0, 1, \cdots, N-1.$

Main lobe width: $\frac{2\omega_s}{N}$ (radian/sec.) or $\frac{4\pi}{N}$ (radian)

Maximum amplitude of side lobe: -13 (dB)

Side lobe roleoff rate: -20 (dB)/decade (-6 (dB)/octave)

Hamming Window:

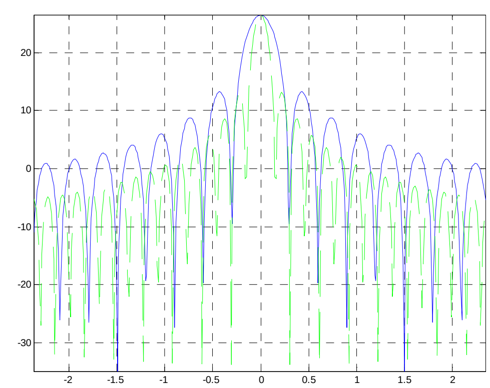
$$w_{hm}(n) = 0.54 - 0.46 \cdot \cos\left(\frac{2\pi n}{N-1}\right), \quad n = 0, 1, \cdots, N-1.$$

Main lobe width: $\frac{4\omega_s}{N}$ (radian/sec.) or $\frac{8\pi}{N}$ (radian)

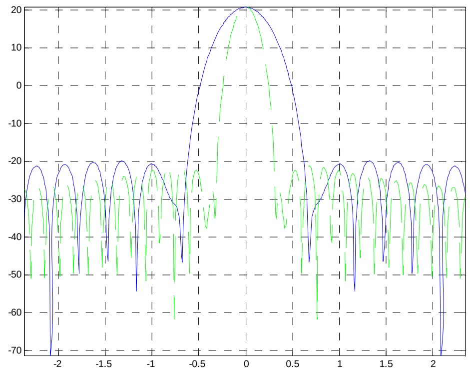
Maximum amplitude of side lobe: -41 (dB)

Side lobe roleoff rate: -20 (dB)/decade (-6 (dB)/octave)

Fourier Transform of Rectangular Window



Fourier Transform of Hamming Window



Blue solid line: N= 21; Green dashed line: N= 41

Hann Window Function:

Hanning Window :

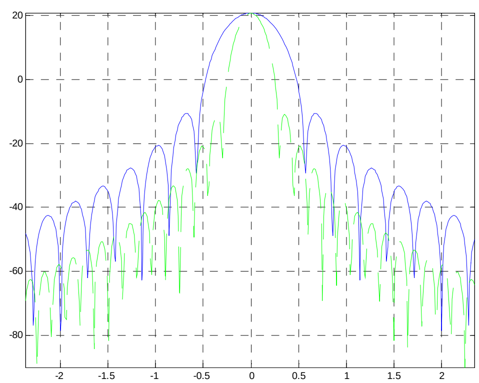
$$w_{hm}(n) = 0.5\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right], \quad n = 0,1,\cdots,N-1.$$

Main lobe width : $\frac{4\omega_s}{N}$ (radian/sec.) or $\frac{8\pi}{N}$ (radian)

Maximum amplitude of side lobe : – 31 (dB)

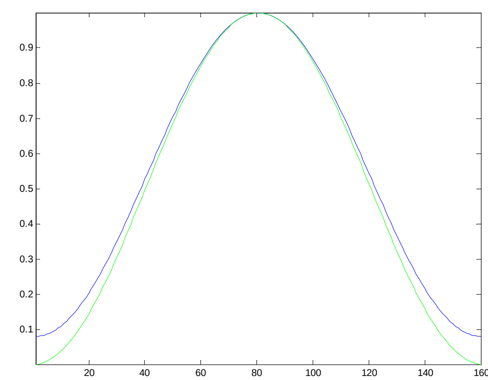
Side lobe roleoff rate : – 60 (dB)/decade (–18 (dB)/octave)

Fourier Transform of Hanning Window



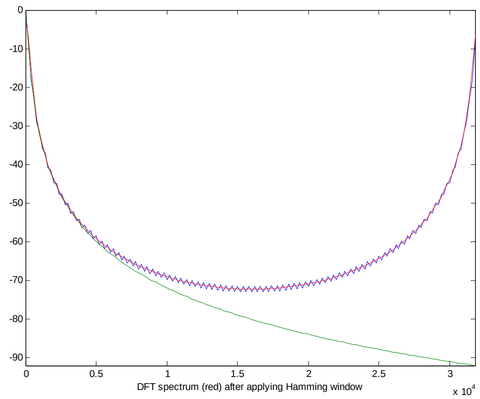
Blue solid line: N= 21; Green dashed line: N= 41

Hanning and Hamming Windows Compared



Blue solid: Hamming window and Green dashed: Hanning window

DFT with Hamming Window



Zero padding to improve frequency resolution

Assumption : $x_p(n)$ contains desired signal energy

Zero padding : $x_e(n) = \{x_p(n)\ 0\ 0 \cdots 0\ 0\}$ to N_e samples

$$X(k) = \sum_{n=0}^{N_e-1} x_e(n) \cdot e^{-j\frac{2\pi}{N_e}k\cdot n} = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j\frac{2\pi}{N_e}k\cdot n}$$

for $k = 0, 1, \cdots, N_e - 1$ with new frequency resolution $\frac{2\pi}{N_e}$.

Matlab : fft(xp, Ne);

Zero padding in the time domain, interpolation in the Fourier domain.

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Fourier Analysis Example

Perform Fourier Analysis of a real valued signal : $x_a(t)$

Signal bandwidth : $f_B = 3$ (KHz)

Signal duration : $D = 100$ (ms) (complete signal energy)

Select sampling frequency : $f_s = 8$ (KHz)

Number of Samples : $N = \frac{D}{T} = 100 \cdot 10^{-3} \cdot 8 \cdot 10^3 = 800$

Frequency resolution : $f_r = \frac{f_s}{N} = 10$ (Hz)

DFT Interpretation : $X_a(\omega) \approx T \cdot X(k \cdot 2\pi \cdot f_r)$

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Fourier Analysis Example continue ….

If the desired frequency resolution is 4 Hz, then

zero padding is needed : $4 = \frac{8000}{N_e} \Rightarrow N_e = 2000$

Require padding 1200 zeros : $N_e - 800 = 1200$

DFT Interpretation : $X_a(\omega) \approx T \cdot X(k \cdot 2\pi \cdot 4)$

Desired DFT dynamic range : 40 dB, then

a Hamming window is needed.

Always apply window before zero padding!

Frequency resolvability reduced to : $\frac{4 \cdot f_s}{N} = 40$ (Hz)

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DFS Example:

$$x_a(t) = 3e^{j(30\pi t - \frac{\pi}{3})} \leftrightarrow X_a(\omega) = 3e^{-j\frac{\pi}{3}} \cdot 2\pi \cdot \delta(\omega - 30\pi)$$

Sampling at $f_s = 120$ (Hz)

$$x(n) = 3e^{j(\frac{30\pi}{120}n - \frac{\pi}{3})} = 3e^{-j\frac{\pi}{3}} e^{j\frac{2\pi}{8}n}, \text{ Period}: N = 8.$$

$$X(k) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi}{8}\cdot nk} = \sum_{n=0}^{7} 3e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{4}n} \cdot e^{-j\frac{2\pi}{8}\cdot nk}$$

$$= 3e^{-j\frac{\pi}{3}} \sum_{n=0}^{7} e^{-j(\frac{2\pi}{8}\cdot k - \frac{\pi}{4})\cdot n} = 3e^{-j\frac{\pi}{3}} \sum_{n=0}^{7} e^{-j(k-1)\frac{\pi}{4}\cdot n}$$

DFS Example:

$$X(k) = \begin{cases} 3e^{-j\frac{\pi}{3}} \sum_{n=0}^{7} e^{-j(k-1)\frac{\pi}{4}\cdot n} = 3e^{-j\frac{\pi}{3}} \sum_{n=0}^{7} e^{-j0} = 3e^{-j\frac{\pi}{3}} \cdot 8, & k = 1 \\ 3e^{-j\frac{\pi}{3}} \dfrac{1 - e^{-j(k-1)\frac{\pi}{4}\cdot 8}}{1 - e^{-j(k-1)\frac{\pi}{4}}} = 3e^{-j\frac{\pi}{3}} \dfrac{1 - e^{-j(k-1)2\pi}}{1 - e^{-j(k-1)\frac{\pi}{4}}} = 0, & k \neq 1 \end{cases}$$

$$= 3e^{-j\frac{\pi}{3}} \cdot 8 \cdot \delta(k-1), \quad k = 0,1,2,\cdots,7$$

$$C_1 = \frac{X(1)}{N} = 3e^{-j\frac{\pi}{3}}$$

$$X_a(\omega) = \frac{2\pi}{N} X(k\frac{\omega_s}{N}) = 3e^{-j\frac{\pi}{3}} \cdot 2\pi \cdot \delta(\omega - 30\pi)$$

DFT Properties:

DFT: $X(k) = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j\frac{2\pi}{N}k\cdot n}, \quad k = 0,1,\cdots,N-1.$

IDFT: $x_p(n) = \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N}k\cdot n}, \quad n = 0,1,\cdots,N-1.$

- If $x_p(n)$ is a real valued signal, then $X(k) = X^*(-k)$,
  i.e., $X(k) = X^*(N-k)$.

- Circular Shift: $x_p(n-m, (\text{modulo } N)) \leftrightarrow X(k) \cdot e^{-j\frac{2\pi}{N}mk}$

$$\sum_{n=0}^{N-1} x_p(n-m) \cdot e^{-j\frac{2\pi}{N}k\cdot n} = \sum_{l=-m}^{N-1-m} x_p(l) \cdot e^{-j\frac{2\pi}{N}k\cdot(l+m)}$$

$$= e^{-j\frac{2\pi}{N}k\cdot m} \cdot \sum_{l=0}^{N-1} x_p(l) \cdot e^{-j\frac{2\pi}{N}k\cdot l} = e^{-j\frac{2\pi}{N}k\cdot m} \cdot X(k)$$

Examples:

Given $x_p(n) = [2 \quad 4 \quad 1 \quad -4]$

Find $X(k)$:

$$X(0) = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j\frac{2\pi}{N}0\cdot n} = \sum_{n=0}^{4-1} x_p(n) \cdot 1 = 3$$

$$X(1) = \sum_{n=0}^{4-1} x_p(n) \cdot e^{-j\frac{2\pi}{4}1\cdot n} = \sum_{n=0}^{3} x_p(n) \cdot (-j)^n = 2 - 4j - 1 + 4j = 1$$

$$X(2) = \sum_{n=0}^{4-1} x_p(n) \cdot e^{-j\frac{2\pi}{4}2\cdot n} = \sum_{n=0}^{3} x_p(n) \cdot (-1)^n$$

$$= \sum_{n=even} x_p(n) - \sum_{n=odd} x_p(n) = (2+1) - (4-4) = 3$$

$\because x_p(n)$ is a real valued signal, then $X(3) = X^*(4-3) = X^*(1) = 1$,

Summery (In place DFT):

If $x_p(n)$ is a real valued signal, then

$X(0)=\sum_{n=0}^{N-1} x_p(n)$ (real valued) and if N is an even number

$X(\frac{N}{2})=\sum_{n=even} x_p(n)-\sum_{n=odd} x_p(n)$ (real valued)

$\because X(N-k)=X^*(k)$

Only $\frac{N}{2}-1$ complex values need to be stored. Total memory locations needed are $2+2(\frac{N}{2}-1)=N$ which is the same number of storge spaces required for the input data sequence.

Linear Convolution Sum:

$$y(n)=x(n)*h(n)$$

$$=\sum_{m=-\infty}^{\infty} x(m)h(n-m)=\sum_{m=-\infty}^{\infty} x(n-m)h(m)$$

S_x, E_x are the starting and ending index for $x(n)$

$L_x=E_x-S_x+1$ is the length of $x(n)$

S_h, E_h are the starting and ending index for $h(n)$

$L_h=E_h-S_h+1$ is the length of $h(n)$

$S_y=S_x+S_h, E_y=E_x+E_h$ for the output $y(n)$

$L_y=L_x+L_h-1$ is the length of the output $y(n)$

Linear Convolution Sum Example:

$x(n)=2^n\left(u(n+1)-u(n-3)\right)\Rightarrow S_x=-1, E_x=2$

$h(n)=[1\quad 3\quad -1]$ with $S_h=2, E_h=4$

$y(n)=x(n)*h(n)$

n	$\cdots$	-2	-1	0	1	2	3	$\cdots$	
$x(m)$				0.5	1	2	4		
$h(n-m)$	-1	3	1						
n	$n-4$		$n-2$		n				

$S_y=-1+2=1\Leftarrow(n-2=-1); E_y=2+4=6\Leftarrow(n-4=2)$

$L_y=4+3-1=6;\quad or\quad L_y=6-1+1=6$

$y(1)=0.5, y(2)=1+3\cdot 0.5=2.5, y(3)=2+3-0.5=4.5$

$y(4)=9, y(5)=10, y(6)=-4.$

Linear Convolution using DFT:

$L_y=L_x+L_h-1=6$, DFT length $N=6\Rightarrow y(n)=IDFT\{Y(k)\}$

Circular Convolution Sum : $x_e(n)*_c h_e(n)\leftrightarrow X_e(k)H_e(k)$

$x_e(n)$ and $h_e(n)$ must be the same length : $0,1,\cdots,N-1.$

$x_e(n)=[0.5\quad 1\quad 2\quad 4\quad 0\quad 0]; h_e(n)=[1\quad 3\quad -1\quad 0\quad 0\quad 0];$

$X_e(k)=[7.5\quad -4-j2.6\quad 3+j0.866\quad -2.5\quad 3-j0.866\quad ?];$

$H_e(k)=[3\quad 3-j1.73\quad -j3.46\quad -3\quad j3.46\quad 3+j1.73];$

$Y(k)=[22.5\quad -16.5-j0.87\quad 3-j10.4\quad 7.5\quad ?\quad ?];$

$y(n)=IDFT\{Y(k)\}=[0.5\quad 2.5\quad 4.5\quad 9\quad 10\quad -4];$