EE 322 First MidTerm

Spring 2000

(1) (a) $\omega_1 = 240\pi$, $\omega_2 = 320\pi$, $\omega_3 = 560\pi$

$\omega_0 = 80\pi$ (MCF)

$x_a(t)$ is periodic and $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{80\pi} = \frac{1}{40}$ (second)

(b)

$$x_a(t) = 2\left(\frac{e^{j(240\pi t+\frac{\pi}{3})} - e^{-j(240\pi t+\frac{\pi}{3})}}{2j}\right) + \left(\frac{e^{j(320\pi t-\frac{\pi}{4})} - e^{-j(320\pi t-\frac{\pi}{4})}}{2j}\right) - \left(\frac{e^{j(560\pi t+1)} + e^{-j(560\pi t+1)}}{2}\right)$$

$$= e^{j(240\pi t+\frac{\pi}{3})} \cdot e^{-j\frac{\pi}{2}} - e^{-j(240\pi t+\frac{\pi}{3})} \cdot e^{-j\frac{\pi}{2}} + \frac{1}{2}e^{-j\frac{\pi}{2}} e^{j(320\pi t-\frac{\pi}{4})} - \frac{1}{2}e^{-j\frac{\pi}{2}} e^{-j(320\pi t-\frac{\pi}{4})} - \frac{1}{2}e^{j(560\pi t+1)} - \frac{1}{2}e^{-j(560\pi t+1)}$$

(1) (b) Continue.

$$= e^{-j\frac{\pi}{6}} \cdot e^{j240\pi t} + e^{j\pi} \cdot e^{-j\frac{\pi}{3}} e^{-j\frac{\pi}{2}} \cdot e^{-j240\pi t}$$

$$+ \frac{1}{2} e^{-j\frac{\pi}{2}} e^{-j\frac{\pi}{4}} e^{j320\pi t} + \frac{1}{2} e^{j\pi} e^{-j\frac{\pi}{2}} e^{+j\frac{\pi}{4}} \cdot e^{-j320\pi t}$$

$$- \frac{1}{2} e^{j} \cdot e^{j560\pi t} - \frac{1}{2} e^{-j} e^{-j560\pi t}$$

$$= e^{-j\frac{\pi}{6}} e^{j240\pi t} + e^{j\frac{\pi}{6}} e^{-j240\pi t}$$

$$+ \frac{1}{2} e^{-j\frac{3\pi}{4}} e^{j320\pi t} + \frac{1}{2} e^{j\frac{3\pi}{4}} e^{-j320\pi t}$$

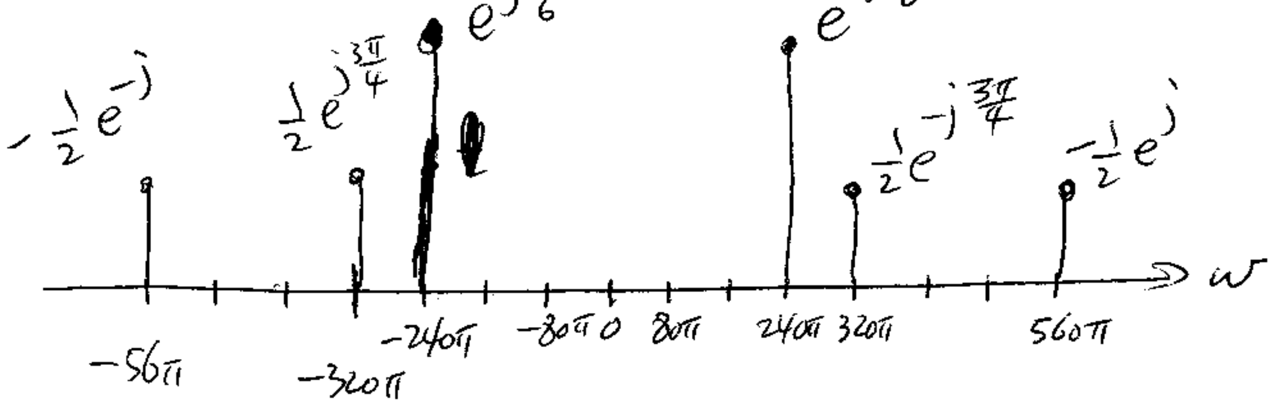
$$- \frac{1}{2} e^{j} e^{j560\pi t} - \frac{1}{2} e^{-j} e^{-j560\pi t}$$

$$\because \omega_0 = 80\pi \quad \therefore \quad D_3 = D_{-3}^* = e^{-j\frac{\pi}{6}}$$

$$D_4 = D_{-4}^* = \frac{1}{2} e^{-j\frac{3\pi}{4}}$$

$$D_7 = D_{-7}^* = -\frac{1}{2} e^{j} = \frac{1}{2} e^{j(1+\pi)}$$

(C)



(d) $x(n) = x_a(t)\Big|_{t=nT} = 2\,\text{Sin}\left(\frac{240\pi}{1120}n + \frac{\pi}{3}\right) + \text{Sin}\left(\frac{320\pi}{1120}n - \frac{\pi}{4}\right) - \text{Cos}\left(\frac{560\pi}{1120}n + 1\right)$

$f_s = 2f_N = 2 \times 560$
$= 1120$ Hz

$$x(n) = 2\,\text{Sin}\left(\frac{3\pi}{14}n + \frac{\pi}{3}\right) + \text{Sin}\left(\frac{2\pi}{7}n - \frac{\pi}{4}\right) - \text{Cos}\left(\frac{\pi}{2}n + 1\right)$$

(e) Rewrite $\hat{\omega}_1 = \frac{2 \cdot 3\pi}{28}$, $\hat{\omega}_2 = \frac{4 \times 2\pi}{28}$, $\hat{\omega}_3 = \frac{7 \times 2\pi}{28}$

$x(n)$ is periodic and Period $\underline{\underline{N}} = 28$ samples.

(f) 28 samples covers 3 period of $\hat{\omega}_1$, i.e., 3 period of $\omega_1 = 240\pi$.
It is exactly one period of 80π sinusoid.

Or. $T_0 = \frac{1}{40}$, $N \cdot T = 28 \cdot \frac{1}{1120} = \frac{1}{40}$

Therefore, $T_0 = NT$ 28 samples covers exactly one period of $x_a(t)$.

(2) $\varphi(t) = 20\pi t^2 + \beta t + \phi$

(i) for $C_a(0) = 0$, requears $\varphi(0) = \phi = \frac{\pi}{2}$

(ii) $\omega(t) = \frac{d\varphi(t)}{dt} = 40\pi t + \beta$

$\omega(0) = 2\pi \cdot 50 = \beta$ for start freqeucy

$\omega(T_2) = 40\pi T_2 + 2\pi 50 = 250 \cdot 2\pi$

for ending freqeucy

$T_2 = \frac{500\pi - 100\pi}{40\pi} = 10$ second.

Solution $\phi = \frac{\pi}{2}$, $\beta = 100\pi$, $T_2 = 10$ sec.

(3) $\because k = 0, 1, \cdots, 255$

$\therefore N - 1 = 255$ $\boxed{N = 256}$ period for $X(k)$ is 256 samples

$\cancel{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}}$

$\because e^{j\frac{l \cdot 2\pi}{N}n} \longleftrightarrow N\delta(k-l)$

$l = 0 \rightarrow 1 \longleftrightarrow N\delta(k)$

Signal component is $x(n) = 2 \longleftrightarrow X(0) = 512 = 2N$ indicate $l = 0$

$$X(2) = X^*(254) = X^*(254-N)$$
$$= X^*(-2) \qquad \because N = 256 \text{ is a period}$$

$$\cos\left(\frac{l\cdot 2\pi}{N} n\right) \longleftrightarrow \frac{N}{2}\left[\delta(k-l) + \delta(k+l)\right]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{e^{j\frac{l\cdot 2\pi}{N}n} + e^{-j\frac{l\cdot 2\pi}{N}n}}{2} \qquad \frac{N}{2}\delta(k-l) + \frac{N}{2}\delta(k+l-N)$$

$$\because X(2) = X^*(-2) = \underbrace{128}_{\frac{N}{2}} e^{j\frac{\pi}{2}}$$

$$\therefore l = 2$$

Signal component:

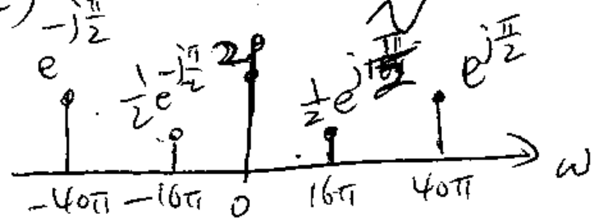
$$x(n) = \frac{e^{j\left(\frac{l\cdot 2\pi}{N}n + \frac{\pi}{2}\right)} + e^{-j\left(\frac{l\cdot 2\pi}{N}n + \frac{\pi}{2}\right)}}{2} = \cos\left(\frac{2\cdot 2\pi}{N}n + \frac{\pi}{2}\right)$$

Similarly, $\because X(5) = X^*(-5) = \underbrace{256}_{N} e^{j\frac{\pi}{2}}$

Signal component is

$$x(n) = 2\cos\left(\frac{5\cdot 2\pi}{N}n + \frac{\pi}{2}\right)$$

(C)



(a) Total $x(n) = 2 + \cos\left(\frac{4\pi}{256}n + \frac{\pi}{2}\right) + 2\cos\left(\frac{10\pi}{256}n + \frac{\pi}{2}\right)$

(b) $x(n)$ has the exactly same period as $X(k)$
$N = 256$ samples.

(c) If $f_s = 1024$, then $\omega_1 = \hat{\omega}_1/T = \hat{\omega}_1 \cdot f_s = \frac{4\pi}{256}\cdot 1024 = 16\pi$

$\omega_2 = \frac{10\pi}{256} f_s = 40\pi \quad x_a(t) = 2 + \cos\left(\frac{4\pi}{256}\cdot f_s \cdot t + \frac{\pi}{2}\right) + 2\cos\left(\frac{10\pi}{256} f_s t + \frac{\pi}{2}\right)$

(d) $x_a(t) = 2 + \cos\left(16\pi t + \frac{\pi}{2}\right) + 2\cos\left(40\pi t + \frac{\pi}{2}\right)$

Similarly $\hat{\omega}_2 \cdot f_s = \frac{10\pi}{256} \cdot 1024 = 40\pi$

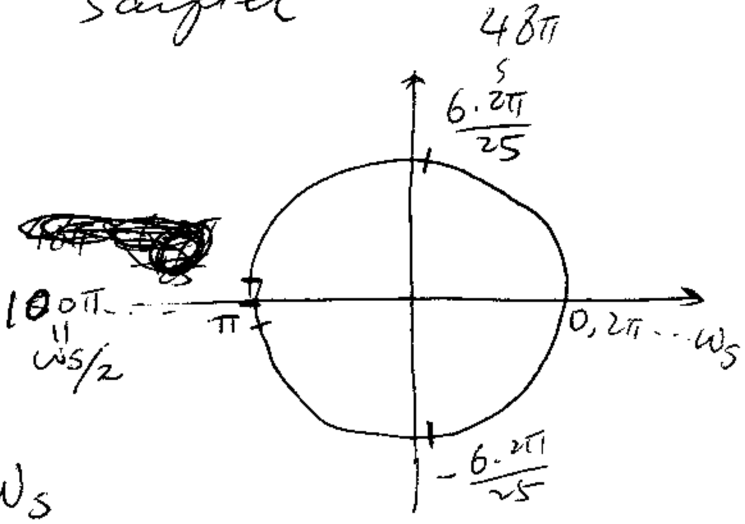
(4) $x_a(t) = 8 \cdot \cos(48\pi t - 2)$, $f_s = 100\,Hz$

(a) $x(n) = 8 \cdot \cos\left(\frac{48\pi n}{100} - 2\right) = 8\cos\left(\frac{6 \cdot 2\pi}{25} n - 2\right)$

$N = 25$ samples

(b) $\hat{\omega}_0 = \frac{6 \cdot 2\pi}{25}$

$\omega_0 = \hat{\omega}_0 \cdot f_s$
$= 48\pi$



$\omega_1 = \omega_0 + \omega_s$
$= 48\pi + 2\pi \cdot f_s = 248\pi$

$\hat{\omega}_1 = \omega_1 \cdot T = 248\pi \cdot \frac{1}{100} = \frac{248}{100}\pi$
$= 2\pi + \frac{48\pi}{100} = \frac{48\pi}{100}$ $\because 2\pi$ is a period

$\boxed{y(n) = 8 \cdot \cos\left(\frac{248\pi}{100} n - 2\right)}$

$\omega_2 = \omega_s - \omega_0 = 200\pi - 48\pi = 152\pi$

$\hat{\omega}_2 = \frac{152\pi}{100} = 2\pi - \frac{48\pi}{100} = \frac{-48\pi}{100}$

$$\boxed{w(n) = 8\cos\left(\frac{152\pi}{100} n + 2\right)}$$

$= 8\cos\left(2\pi - \frac{48\pi}{100} n + 2\right) = 8\cos\left(-\left(\frac{48\pi}{100} n - 2\right)\right)$

$= 8\cos\left(\frac{48\pi}{100} n - 2\right)$

(7)

(5) $x_a(t) = Re\left\{5e^{(j20\pi t-2)} + e^{j(20\pi t - \frac{\pi}{2}+1)} - 3e^{j(20\pi t-0.5)}\right\}$

$= Re\left\{\left(5e^{-j2} + e^{-j(\frac{\pi}{2}-1)} - 3e^{-j0.5}\right)e^{j20\pi t}\right\}$

$= Re\left\{(5.32e^{-j2.4})e^{j20\pi t}\right\}$

$= Re\left\{5.32e^{j(20\pi t-2.4)}\right\} = 5.32\cos(20\pi t-2.4)$

$A = 5.32, \quad \phi = -2.4$

(6) $x_a(t) = \frac{1}{2}\cos(800\pi t) + \frac{1}{2}\cos(1200\pi t)$

(a) $\frac{\omega_1}{\omega_2} = \frac{800\pi}{1200\pi} = \frac{2\cdot 400\pi}{3\cdot 400\pi}$

$\omega_0 = 400\pi$

(b) $T_0 = \frac{1}{200}\left(=\frac{2\pi}{\omega_0}\right)$ $T = \frac{T_0}{5} = \frac{1}{1000}$ 5 samples/period

$f_s = \frac{1}{T} = 1000\,Hz$

(c) The Nyquist frequency of $x_a(t)$ is 1200Hz. $f_s < f_N$. We expect aliasing error in the sampled signal.

(d) $x(n) = \frac{1}{2}\cos\left(\frac{800\pi}{1000}n\right) + \frac{1}{2}\cos\left(\frac{1200\pi}{1000}n\right)$

$= \frac{1}{2}\cos\left(\frac{4\pi}{5}n\right) + \frac{1}{2}\cos\left(\frac{6\pi}{5}n\right)$

$= \frac{1}{2}\cos\left(\frac{4\pi}{5}n\right) + \frac{1}{2}\cos\left(\left(2\pi - \frac{6\pi}{5}\right)n\right)$

$\frac{6\pi}{5} > \pi$ — folding frequency

$= \cos\left(\frac{4\pi}{5}n\right)$

$\hat{\omega}_0 = \frac{4\pi}{5}$ $\quad \omega_0 = \hat{\omega}_0 \cdot f_s = \frac{4\pi}{5} \cdot 1000$

$= 800\pi$

$x_a(t) = \cos(800\pi)$ is the reconstructed signal.

(e) $T = \frac{T_0}{7} = \frac{1}{1400}$ $\quad f_s = 1400 > f_N = 1200$

$x(n) = \frac{1}{2}\cos\left(\frac{800\pi}{1400}n\right) + \frac{1}{2}\cos\left(\frac{1200\pi}{1400}n\right)$

$= \frac{1}{2}\cos\left(\frac{4\pi}{7}n\right) + \frac{1}{2}\cos\left(\frac{6\pi}{7}n\right)$

Both $\frac{4\pi}{7} < \pi$ and $\frac{6\pi}{7} < \pi$

reconstructed: $\frac{4\pi}{7} \cdot f_s = 800\pi$

$\frac{6\pi}{7} \cdot f_s = 1200\pi$

No aliasing error.