CHAP. 14: SYSTEMS OF PARTICLES

PRINCIPLES DISCUSSED IN THIS CHAPTER WILL APPLY TO A SYSTEM OF INDIVIDUAL PARTICLES AND TO A SYSTEM OF RIGIDLY CONNECTED PARTICLES: A RIGID BODY

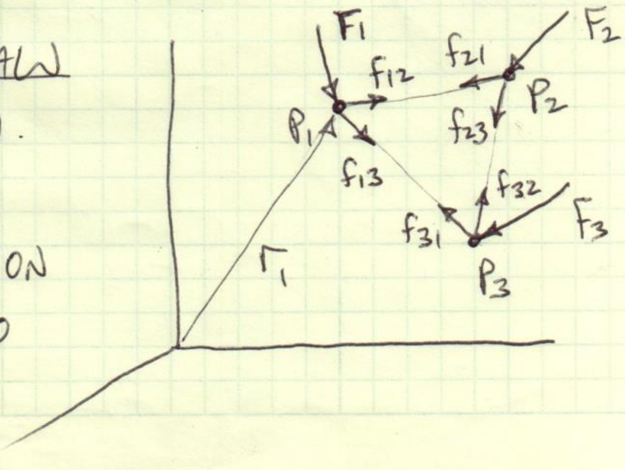
TOPICS:

- NEWTON'S 2ND LAW APPLIED TO A SYSTEM OF PARTICLES
- SUM OF RESULTANT EXTERNAL FORCES = RATE OF CHANGE OF THE TOTAL LINEAR MOMENTUM OF THE SYSTEM
- SUM OF RESULTANT MOMENTS OF EXTERNAL FORCES = RATE OF CHANGE OF THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM
- MASS CENTER OF THE SYSTEM
- CONSERVATION OF LINEAR AND ANGULAR MOMENTUM
- PRINCIPLES OF WORK/ENERGY AND IMPULSE/MOMENTUM TO A SYSTEM OF PARTICLES

<u>NEWTON'S 2ND LAW</u>

$\vec{F}_i$ = EXTERNAL FORCES ON PARTICLE i

$\vec{f}_{ij}$ = INTERNAL FORCE ON PARTICLE i DUE TO PARTICLE j



NEWTON'S 2ND LAW FOR EACH PARTICLE:

$$\vec{F}_1 + \vec{f}_{12} + \vec{f}_{13} = m_1 \vec{a}_1$$

$$\vec{F}_2 + \vec{f}_{21} + \vec{f}_{23} = m_2 \vec{a}_2$$

$$\vec{F}_3 + \vec{f}_{31} + \vec{f}_{32} = m_3 \vec{a}_3$$

ADD 3 VECTOR EQUATIONS:

$$(\vec{F}_1 + \vec{F}_2 + \vec{F}_3) + (\vec{f}_{12} + \vec{f}_{21} + \vec{f}_{13} + \vec{f}_{31} + \vec{f}_{23} + \vec{f}_{32})$$

$$= (m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3)$$

NOTE THAT $\vec{f}_{12}$ AND $\vec{f}_{21}$ ARE EQUAL AND OPPOSITE, SO $\vec{f}_{12} + \vec{f}_{21} = 0$. NEWTON'S 2ND LAW REDUCES TO

$$\boxed{\sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} m_i \vec{a}_i}$$

TAKING THE CROSS PRODUCT OF POSITION VECTOR $\vec{r}_1$ INTO THE FIRST EQUATION GIVES

$$\vec{r}_1 \times \vec{F}_1 + \vec{r}_1 \times \vec{f}_{12} + \vec{r}_1 \times \vec{f}_{13} = \vec{r}_1 \times m_1 \vec{a}_1$$

SIMILARLY,

$$\vec{r}_2 \times \vec{F}_2 + \vec{r}_2 \times \vec{f}_{21} + \vec{r}_2 \times \vec{f}_{23} = \vec{r}_2 \times m_2 \vec{a}_2$$

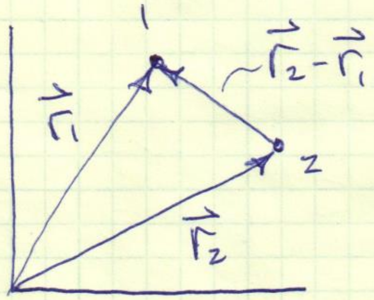
$$\vec{r}_3 \times \vec{F}_3 + \vec{r}_3 \times \vec{f}_{31} + \vec{r}_3 \times \vec{f}_{32} = \vec{r}_3 \times m_3 \vec{a}_3$$

ADD THESE 3 VECTORS EQUATIONS, THEN NOTE THAT

$$\vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} = \vec{r}_1 \times \vec{f}_{12} + \vec{r}_1 \times \vec{f}_{21} + \vec{r}_2 \times \vec{f}_{21} - \vec{r}_1 \times \vec{f}_{21}$$

$$= \vec{r}_1 \times (\vec{f}_{12} + \vec{f}_{21}) + (\vec{r}_2 - \vec{r}_1) \times \vec{f}_{21}$$

$$\vec{f}_{12} + \vec{f}_{21} = 0$$



SINCE $\vec{r}_2 - \vec{r}_1$ IS COLLINEAR WITH $\vec{f}_{21}$,

$$(\vec{r}_2 - \vec{r}_1) \times \vec{f}_{21} = 0$$

THEREFORE,

$$\vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} = 0$$

$$\vec{r}_2 \times \vec{f}_{23} + \vec{r}_3 \times \vec{f}_{32} = 0$$

$$\vec{r}_3 \times \vec{f}_{31} + \vec{r}_1 \times \vec{f}_{13} = 0$$

$$\cancel{\vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} = 0}$$

~~SINCE THEY CREATE EQUAL AND OPPOSITE MOMENT VECTORS ABOUT THE ORIGIN.~~ THE RESULTING MOMENT VECTOR EQUATION REDUCES TO

$$\boxed{\sum_{i=1}^{n} (\vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{a}_i) = \sum_{i=1}^{n} \vec{M}_0}$$

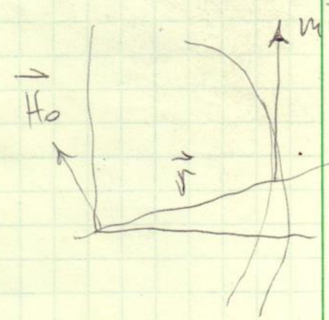
LINEAR AND ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

$$\vec{L} = \sum_{i=1}^{n} m_i \vec{v}_i$$ LINEAR MOMENTUM ($\vec{L} = m\vec{v}$ for a particle)

$$\vec{H}_0 = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i)$$ ANGULAR MOMENTUM ($\vec{H}_0 = \vec{r} \times m\vec{v}$ for a particle)

DIFFERENTIATE BOTH EQUATIONS W.R.T. TIME GIVES

$$\boxed{\dot{\vec{L}} = \sum m_i \dot{\vec{v}}_i = \sum m_i \vec{a}_i}$$



$$\dot{\vec{H}}_0 = \sum (\dot{\vec{r}}_i \times m_i \vec{v}_i) + \sum (\vec{r}_i \times m_i \dot{\vec{v}}_i)$$

$$\dot{\vec{H}}_0 = \sum (\vec{v}_i \times m_i \vec{v}_i) + \sum (\vec{r}_i \times m_i \vec{a}_i)$$

$$\vec{v}_i \times m_i \vec{v}_i = 0$$ COLLINEAR

$$\boxed{\dot{\vec{H}}_0 = \sum (\vec{r}_i \times m_i \vec{a}_i)}$$

FROM PREVIOUS EQUATIONS,

$$\boxed{\sum \vec{F}_i = \sum m_i \vec{a}_i = \dot{\vec{L}}}$$

SUM OF EXTERNAL FORCES = RATE OF CHANGE OF LINEAR MOMENTUM

$$\boxed{\sum \vec{M}_0 = \sum (\vec{r}_i \times m_i \vec{a}_i) = \dot{\vec{H}}_0}$$

SUM OF MOMENTS OF EXTERNAL FORCES = RATE OF CHANGE OF ANGULAR MOMENTUM

MASS CENTER OF A SYSTEM OF PARTICLES

CENTER OF GRAVITY: $W_T \bar{x} = \sum W_i \bar{x}_i$

CENTROID OF AREA: $A_T \bar{x} = \sum A_i \bar{x}_i$

CENTER OF MASS IS LOCATED AT POINT G

$$m\bar{\vec{r}} = \sum m_i \vec{r}_i$$

$m = \sum m_i$ = TOTAL SYSTEM MASS

$\bar{\vec{r}}$ = POSITION VECTOR TO MASS CENTER

m_i = INDIVIDUAL PARTICLE MASS

$\vec{r}_i$ = POSITION VECTOR TO PARTICLE

DIFFERENTIATE W.R.T. TIME:

$$m\dot{\bar{\vec{r}}} = \sum m_i \dot{\vec{r}}_i$$

$$m\bar{\vec{v}} = \sum m_i \vec{v}_i$$

$\vec{v}$ = VELOCITY OF MASS CENTER G

FROM BEFORE, $\vec{L} = \sum m_i \vec{v}_i = m\vec{v}$

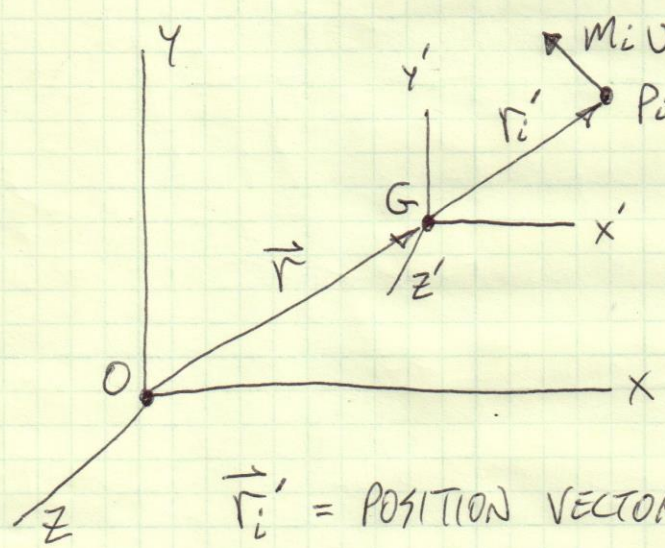
DIFF. W.R.T. TIME GIVES

$\dot{\vec{L}} = m\vec{a}$ OR $\boxed{\sum \vec{F} = m\vec{a}}$

$\vec{a}$ = ACCELERATION OF MASS CENTER G

THE MASS CENTER OF A SYSTEM OF PARTICLES MOVES AS IF THE ENTIRE MASS OF THE SYSTEM AND ALL THE EXTERNAL FORCES WERE CONCENTRATED AT THAT POINT.

CONSIDER A TRANSLATING FRAME OF REFERENCE ATTACHED TO THE MASS CENTER G:



$\vec{r}_i'$ = POSITION VECTOR RELATIVE TO G AXES

THE ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES RELATIVE TO G IS

$$\vec{H}'_G = \Sigma(\vec{r}_i' \times m_i \vec{v}_i')$$

DIFF. W.R.T. TIME:

$$\dot{\vec{H}}'_G = \Sigma(\vec{r}_i' \times m_i \vec{a}_i')$$

$\vec{a}_i'$ = ACCELERATION RELATIVE TO G AXES

THE ACCELERATION OF POINT P_i RELATIVE TO THE FIXED O AXES IS

$$\vec{a}_i = \vec{a} + \vec{a}_i'$$

$\vec{a}$ = ACCEL OF G RELATIVE TO O

$$\dot{\vec{H}}'_G = \Sigma[\vec{r}_i' \times m_i(\vec{a}_i - \vec{a})]$$

$$= \Sigma(\vec{r}_i' \times m_i \vec{a}_i) - \Sigma(\vec{r}_i' \times m_i \vec{a})$$

$$\Sigma(\vec{r}_i' \times m_i \vec{a}) = \Sigma[(m_i \vec{r}_i') \times \vec{a}] = \Sigma(m_i \vec{r}_i') \times \vec{a}$$

FROM BEFORE, $m\vec{r}' = \Sigma m_i \vec{r}_i'$

BUT $\vec{r}' = 0$ BECAUSE THE DISTANCE TO THE MASS CENTER FROM G IS ZERO.

$$\boxed{\dot{\vec{H}}_G' = \Sigma(\vec{r}_i' \times m_i \vec{a}_i) = \Sigma \vec{M}_G}$$

SUM OF THE MOMENTS ABOUT G OF THE EXTERNAL FORCES = RATE OF CHANGE OF ANGULAR MOMENTUM RELATIVE TO THE TRANSLATING G AXES.

WITH RESPECT TO THE FIXED O AXES, THE ANGULAR MOMENTUM OF THE SYSTEM IS

$$\vec{H}_G = \Sigma(\vec{r}_i' \times m_i \vec{v}_i)$$

THE VELOCITY RELATIVE TO THE O AXES IS

$$\vec{v}_i = \vec{v} + \vec{v}_i'$$

SUBSTITUTING GIVES

$\Sigma(m_i \vec{r}_i') = m\vec{r}' = 0$

BECAUSE THE DISTANCE TO THE MASS CENTER IS ZERO.

$$\vec{H}_G = \Sigma[\vec{r}_i' \times m_i(\vec{v} + \vec{v}_i')]$$

$$= \Sigma(m_i \vec{r}_i') \times \vec{v} + \Sigma(\vec{r}_i' \times m_i \vec{v}_i')$$

$$\vec{H}_G = \Sigma(\vec{r}_i' + m_i \vec{v}_i') = \vec{H}_G' \cancel{= \Sigma(\vec{r}_i' \times m_i \vec{v}_i')}$$

THE ANGULAR MOMENTUM OF THE SYSTEM OF PARTICLES MEASURED FROM THE FIXED O AXES IS EQUAL TO THAT MEASURED FROM THE TRANSLATING G AXES. THEREFORE,

$$\boxed{\Sigma \vec{M}_G = \dot{\vec{H}}_G = \dot{\vec{H}}_G'}$$

ANGULAR MOMENTUM CAN BE COMPUTED USING EITHER REFERENCE FRAME. THESE IMPORTANT RESULTS WILL BE USED IN THE ANALYSIS OF RIGID BODIES.

CONSERVATION OF MOMENTUM FOR A SYSTEM OF PARTICLES

$$\Sigma \vec{F} = \dot{\vec{L}} \quad , \quad \Sigma \vec{M}_0 = \dot{\vec{H}}_0$$

IF NO EXTERNAL FORCES ACT ON THE PARTICLES,

$$\dot{\vec{L}} = 0 \quad \text{AND} \quad \dot{\vec{H}}_0 = 0 \quad \underline{\text{OR}}$$

$$\vec{L} = \text{CONSTANT} \quad \text{AND} \quad \vec{H}_0 = \text{CONSTANT}$$

KINETIC ENERGY OF A SYSTEM OF PARTICLES

$$(\vec{v}_i = \bar{v} + \vec{v}_i')$$

$$T = \frac{1}{2} \Sigma m_i v_i^2 = \frac{1}{2} m |\bar{v}|^2 + \frac{1}{2} \Sigma m_i |\vec{v}_i'|^2$$

(K.E. OF MASS CENTER) + (K.E. RELATIVE TO G AXES)

PRINCIPLE OF WORK AND ENERGY

$$T_1 + U_{1-2} = T_2$$

T_1 = INITIAL K.E. OF SYSTEM OF PARTICLES

T_2 = FINAL K.E., U_{1-2} = WORK DONE

IF ALL FORCES ARE CONSERVATIVE, (NO FRICTION)

$$T_1 + V_1 = T_2 + V_2$$

V = POTENTIAL ENERGY OF THE PARTICLES

PRINCIPLE OF IMPULSE AND MOMENTUM

$$\Sigma \vec{F} = \dot{\vec{L}} \quad , \quad \Sigma \vec{M}_0 = \dot{\vec{H}}_0$$

INTEGRATE BOTH EQUATIONS W.R.T. TIME GIVES

$$\boxed{\Sigma \int_{t1}^{t2} \vec{F}\, dt = \vec{L}_2 - \vec{L}_1}$$

SUM OF LINEAR IMPULSES = CHANGE IN LINEAR MOMENTUM OF THE SYSTEM

$$\boxed{\Sigma \int_{t1}^{t2} \vec{M}_0\, dt = (\vec{H}_0)_2 - (\vec{H}_0)_1}$$

SUM OF ANGULAR IMPULSES = CHANGE IN ANGULAR MOMENTUM OF THE SYSTEM

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