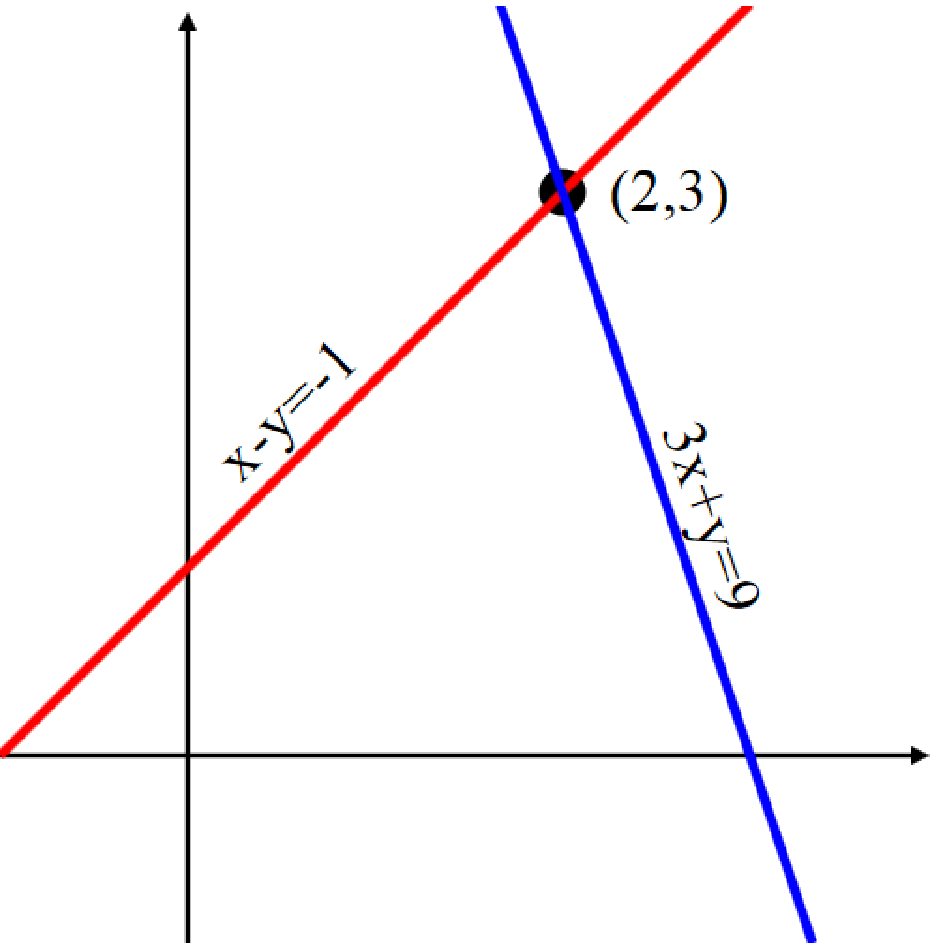
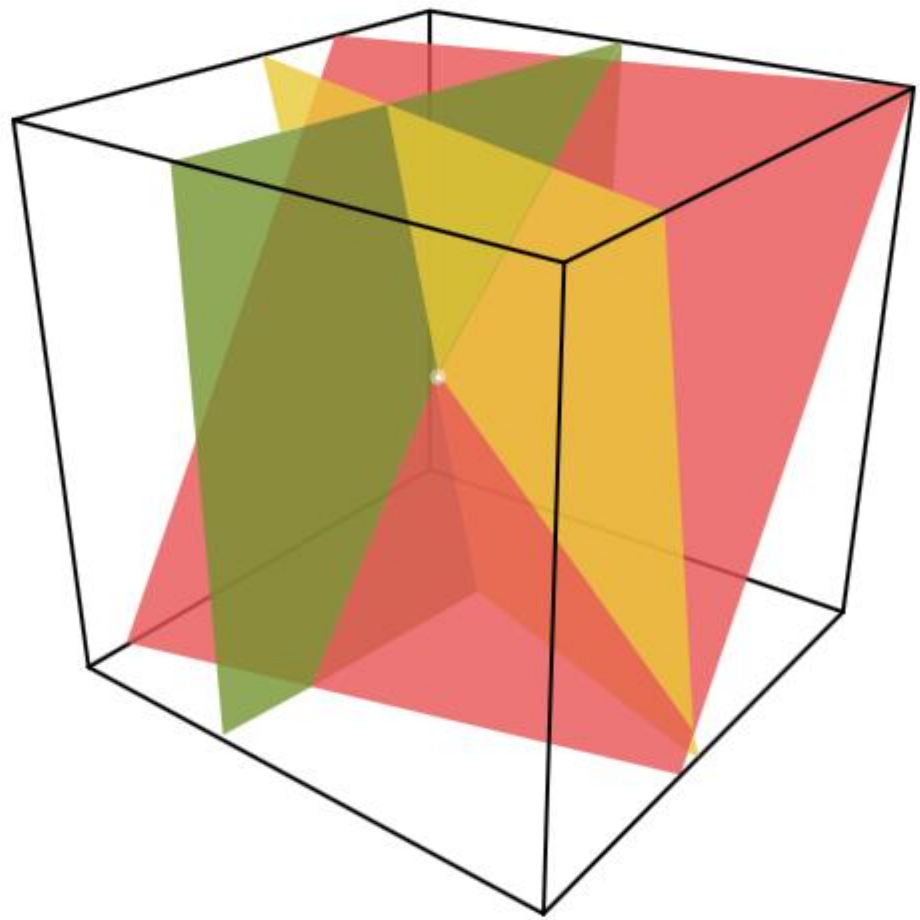
Chapter 8: Linear Algebraic Equations

- Matrix Methods for Linear Equations
- Uniqueness and Existence of Solutions
- Under-Determined Systems
- Over-Determined Systems

Linear Algebraic Equations

For 2 Equations and 2 Unknowns, the solution is the intersection of the two lines. For 3 Equations and 3 Unknowns, the solution is the intersection of three planes.





Matrix Methods for Linear Systems of Equations

The simplest system of linear equations is:

$$ax + by = c$$
$$dx + ey = f$$

Two equations and two unknowns (all coefficients are known). Can be solved by substitution, row reduction, Kramer's Rule. Cast the system in Vector Form:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} c \\ f \end{Bmatrix}$$

Matrix*Column Vector = Column Vector

$$A \cdot z = B$$

Matrix Methods for Linear Systems of Equations

$$A \cdot z = B$$

Solve for solution vector z by multiplying both sides by A^{-1} (A Inverse [Matrix]):

$$A^{-1} \cdot (A \cdot z) = A^{-1} \cdot (B)$$

$$\text{LHS: } A^{-1} \cdot (A \cdot z) = (A^{-1} \cdot A) \cdot z = I \cdot z = z$$

$$A^{-1} \cdot A = I \ \text{ (Identity Matrix): } \ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$z = A^{-1} \cdot B$$

where

$$A^{-1} = \left(\frac{1}{ae - bd}\right) \cdot \begin{bmatrix} e & -b \\ -d & a \end{bmatrix}$$

Matrix Methods for Linear Systems of Equations

A^{-1} Exists if the Determinant of Matrix A is both Square and Nonsingular:

$$|A| = \det(A) = ae - bd \neq 0$$

For a given linear system of equations, there may be:

- One Unique Solution
- An Infinite Number of Solutions
- No Solution

Use the Rank of Matrix A and the Rank of the Augmented Matrix $[A\ B]$ to Establish the Uniqueness and Existence of the Solution.

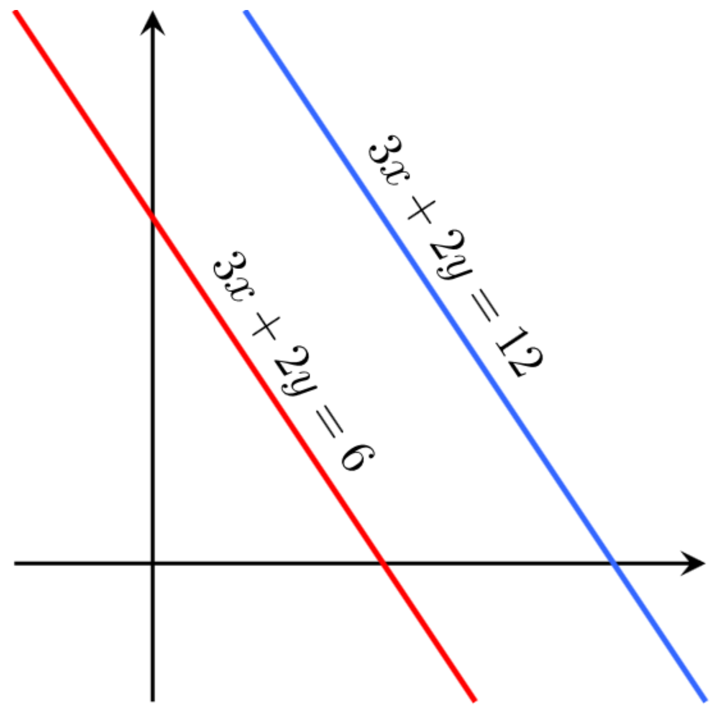
Rank_A = rank(A)
Rank_AB = rank([A B])

Uniqueness and Existence of the Solution

If Rank_A ≠ Rank_AB, a Unique Solution Does Not Exist

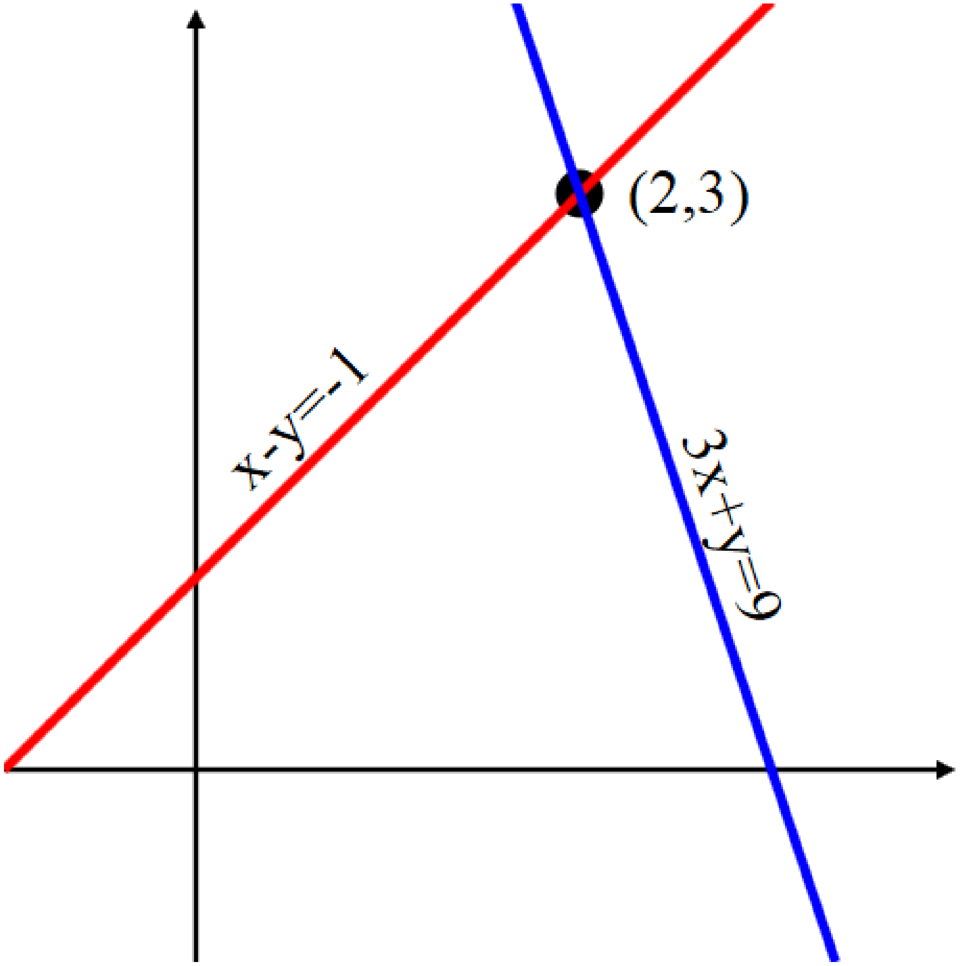
If Rank_A = Rank_AB = Number of Unknowns, a Unique Solution Exists

If Rank_A = Rank_AB but Rank_A ≠ Number of Unknowns, an Infinite Number of Solutions Exist



Uniqueness and Existence of the Solution

```
% One Unique Solution:
% x-y = -1, 3x+y = 9
A = [1 -1; 3 1]
B = [-1; 9]
AB = [A B]
rank_A = rank(A)
rank_AB = rank(AB)
Det_A = det(A)
Inv_A = inv(A)
z = Inv_A*B
```



Matrix Methods for Linear Systems of Equations

MATLAB has Several Methods of Solving Systems of Linear Equations:

- Matrix Inverse: `z = inv(A)*B` (Calculates the Inverse of A.)
- Left Division: `z = A\B` (Uses Gauss Elimination to Solve for z for Over-Determined Systems.)
- Pseudo-Inverse: `z = pinv(A)*B` (Used When There are More Unknowns than Equations: Under-Determined Systems. This Method Provides One Solution, but Not All Solutions.)
- Row-Reduced Echelon Form: `rref([A  B])` (Also Used for Under-Determined Systems. This Method Provides a Set of Equations That Can be Used to Find All Solutions.)

Under-Determined Systems

Not Enough Information to Determine all of the Unknowns (Infinite Number of Solutions)

Case 1: Fewer Equations than Unknowns:

$$x + 3y - 5z = 7$$
$$-8x - 10y + 4z = 28$$

Case 2: Two Equations are Not Independent:

$$x + 3y - 5z = 7$$
$$-8x - 10y + 4z = 28$$
$$-16x - 20y + 8z = 56$$

(Equation 3) = 2*(Equation 2)

Solve Under-Determined Systems using Pseudo-Inverse `pinv`, which finds one solution by minimizing the Euclidean Norm, or find all possible solutions by using the `rref` method (Row-Reduced Echelon Form).

Under-Determined Systems

```
% Under-Determined System:
% Inverse Method
% x+3y-5z = 7, -8x-10y+4z = 28
A = [1 3 -5; -8 -10 4]
B = [7; 28]
AB = [A B]
rank_A = rank(A)
rank_AB = rank(AB)
Det_A = det(A)
Inv_A = inv(A)
z = Inv_A*B
```

Under-Determined Systems

```
% Under-Determined System:
% Pseudo-Inverse Method
% x+3y-5z = 7, -8x-10y+4z = 28
A = [1 3 -5; -8 -10 4]
B = [7; 28]
AB = [A B]
rank_A = rank(A)
rank_AB = rank(AB)
z = pinv(A)*B
```

Row-Reduced Echelon Form

$$x + 3y - 5z = 7$$
$$-8x - 10y + 4z = 28$$

Replace (Equation 2) by 8*(Equation 1) + (Equation 2):

$$x + 3y - 5z = 7$$
$$0x + 14y - 36z = 84$$

Divide (Equation 2) by 14:

$$x + 3y - 5z = 7$$
$$0x + y - \frac{36}{14}z = 6$$

Replace (Equation 1) by −3*(Equation 2) + (Equation 1):

$$x + 0y + 2.714z = -11$$
$$0x + y - \frac{36}{14}z = 6$$

Infinite Number of Solutions because z can be any number:

$$x + 2.714z = -11$$
$$y - \frac{36}{14}z = 6$$

Under-Determined Systems

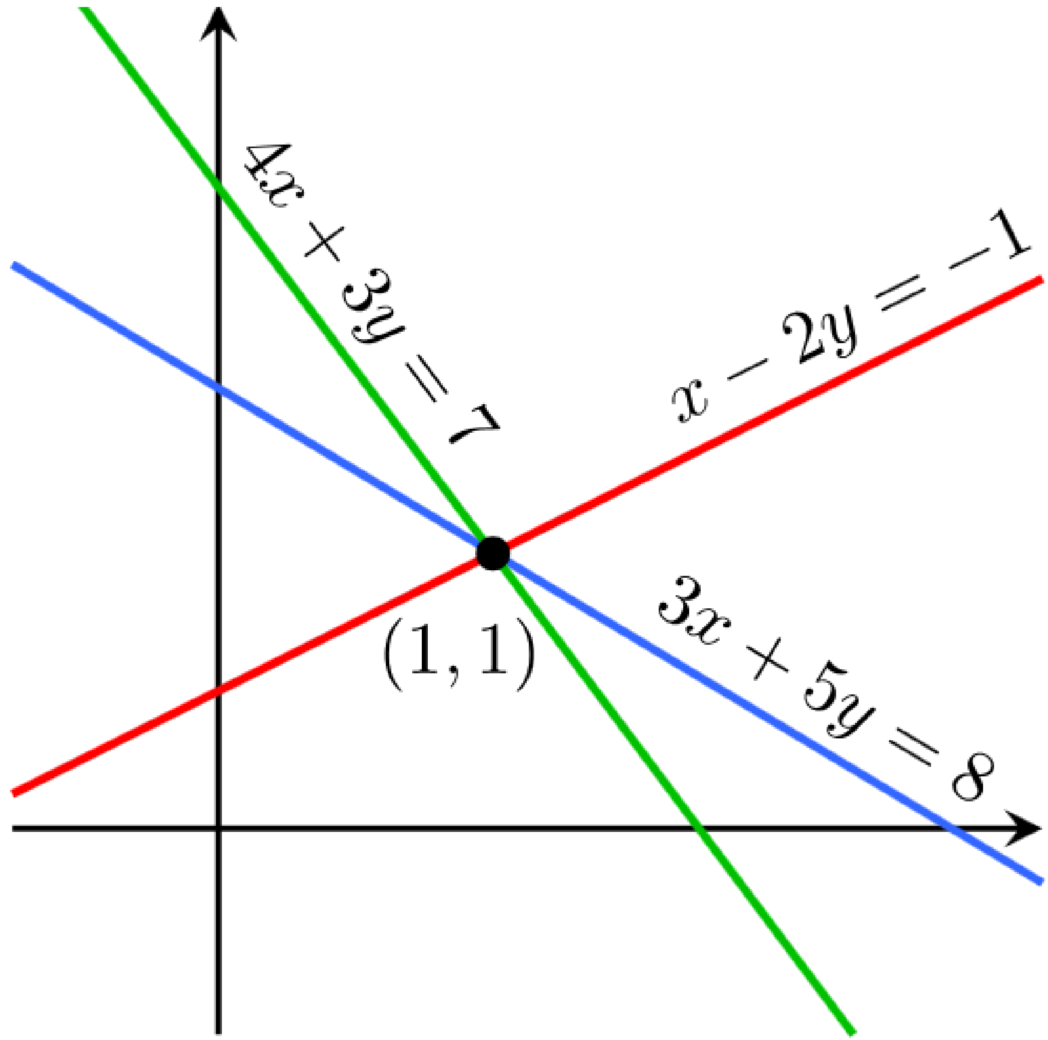
```
% Under-Determined System:
% Row-Reduced Echelon Method
% x+3y-5z = 7, -8x-10y+4z = 28
A = [1 3 -5; -8 -10 4]
B = [7; 28]
AB = [A B]
rank_A = rank(A)
rank_AB = rank(AB)
z = rref([A,B])
```

Over-Determined Systems

More Independent Equations than Unknowns.

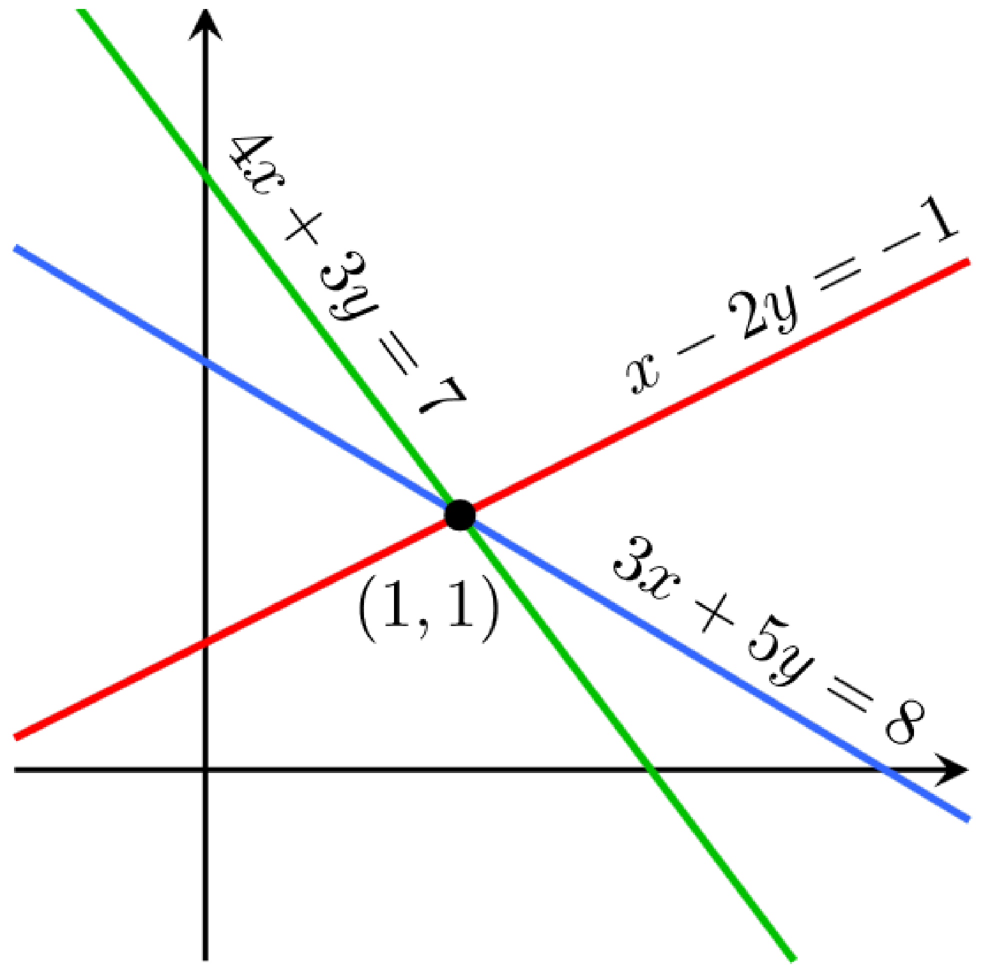
If Rank_A = Rank_AB, A Unique Solution Exists: Use Left Division Method to find the Solution: `z = A\B`

If Rank_A ≠ Rank_AB, No Solution Exists: The Left Division Method gives a Least-Squares Solution, NOT an Exact Solution.



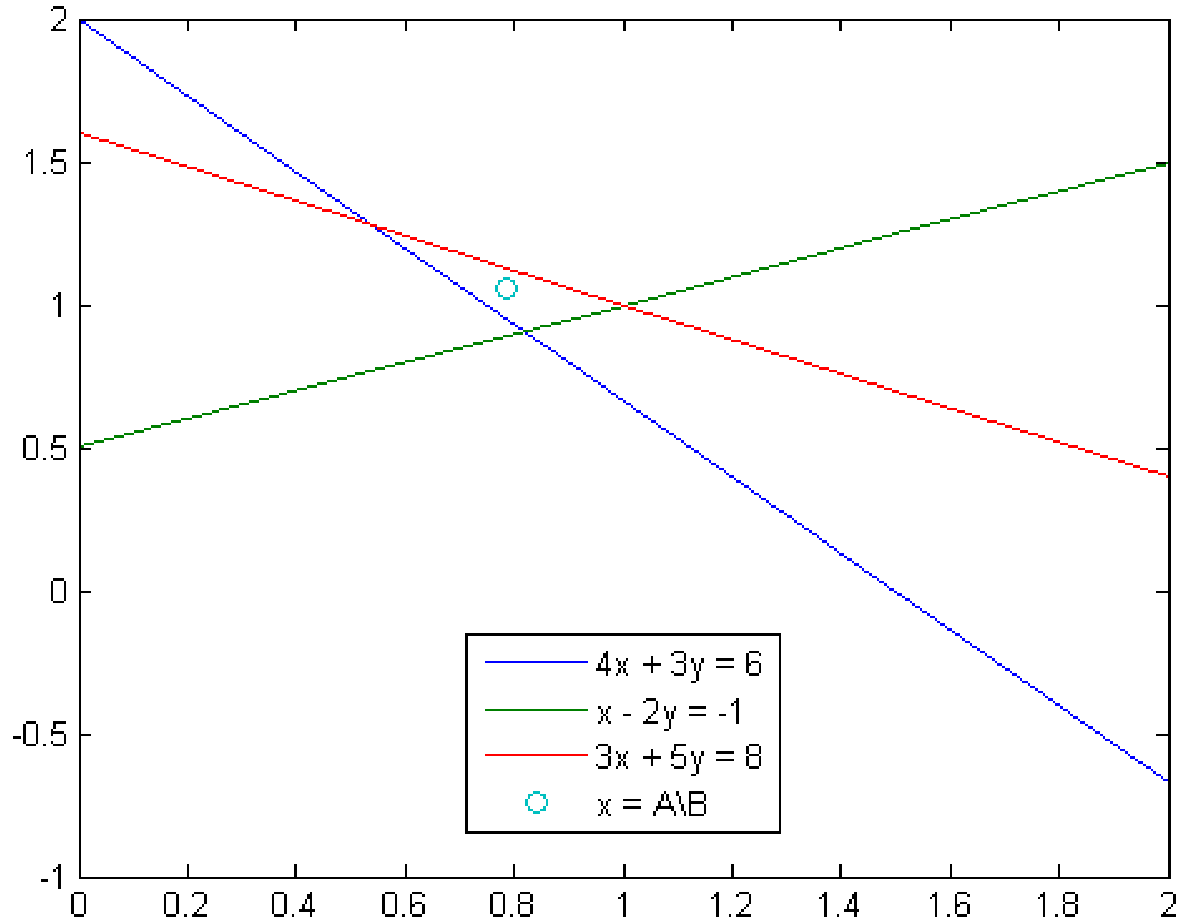
Over-Determined Systems

```
% Over-Determined System:
% 4x+3y = 7; x-2y = -1; 3x+5y = 8
A = [4 3; 1 -2; 3 5]
B = [7; -1; 8]
AB = [A B]
rank_A = rank(A)
rank_AB = rank(AB)
z = A\B
```



Over-Determined Systems

```
% Over-Determined System:
% 4x+3y = 6; x-2y = -1; 3x+5y = 8
A = [4 3; 1 -2; 3 5]
B = [6; -1; 8]
AB = [A B]
rank_A = rank(A)
rank_AB = rank(AB)
z = A\B
```



Problem 8.2:

2.* *a.* Solve the following matrix equation for the matrix **C**.

$$\mathbf{A}(\mathbf{B}\mathbf{C} + \mathbf{A}) = \mathbf{B}$$

b. Evaluate the solution obtained in part *a* for the case

$$\mathbf{A} = \begin{bmatrix} 7 & 9 \\ -2 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & -3 \\ 7 & 6 \end{bmatrix}$$

$$A(BC + A) = B$$
$$A^{-1}A(BC + A) = A^{-1}B$$
$$BC + A = A^{-1}B$$
$$BC = A^{-1}B - A$$
$$B^{-1}BC = B^{-1}(A^{-1}B - A)$$
$$C = B^{-1}(A^{-1}B - A)$$

```
A =

     7     9
    -2     4


B =

     4    -3
     7     6


C =

   -0.8536   -1.6058
    1.5357    1.3372
```

Problem 8.4:

The circuit shown in Figure P4 has five resistances and one applied voltage. Kirchhoff's voltage law applied to each loop in the circuit shown gives

$$v - R_2 i_2 - R_4 i_4 = 0$$
$$-R_2 i_2 + R_1 i_1 + R_3 i_3 = 0$$
$$-R_4 i_4 - R_3 i_3 + R_5 i_5 = 0$$

Conservation of charge applied at each node in the circuit gives

$$i_6 = i_1 + i_2$$
$$i_2 + i_3 = i_4$$
$$i_1 = i_3 + i_5$$
$$i_4 + i_5 = i_6$$

a) Write a MATLAB script file that uses given values of the applied voltage v and the values of the five resistances and solves for the six currents.
b) Use the program developed in part *a*) to find the currents for the case where $R_1 = 1\ \text{k}\Omega$, $R_2 = 5\ \text{k}\Omega$, $R_3 = 2\ \text{k}\Omega$, $R_4 = 10\ \text{k}\Omega$, $R_5 = 5\ \text{k}\Omega$, and $v = 100\ V$ $(1\ \text{k}\Omega = 1000\ \Omega)$.

Problem 8.4:

$$v - R_2 i_2 - R_4 i_4 = 0$$

$$-R_1 i_1 - R_3 i_3 + R_2 i_2 = 0$$

$$-R_5 i_5 + R_4 i_4 + R_3 i_3 = 0$$

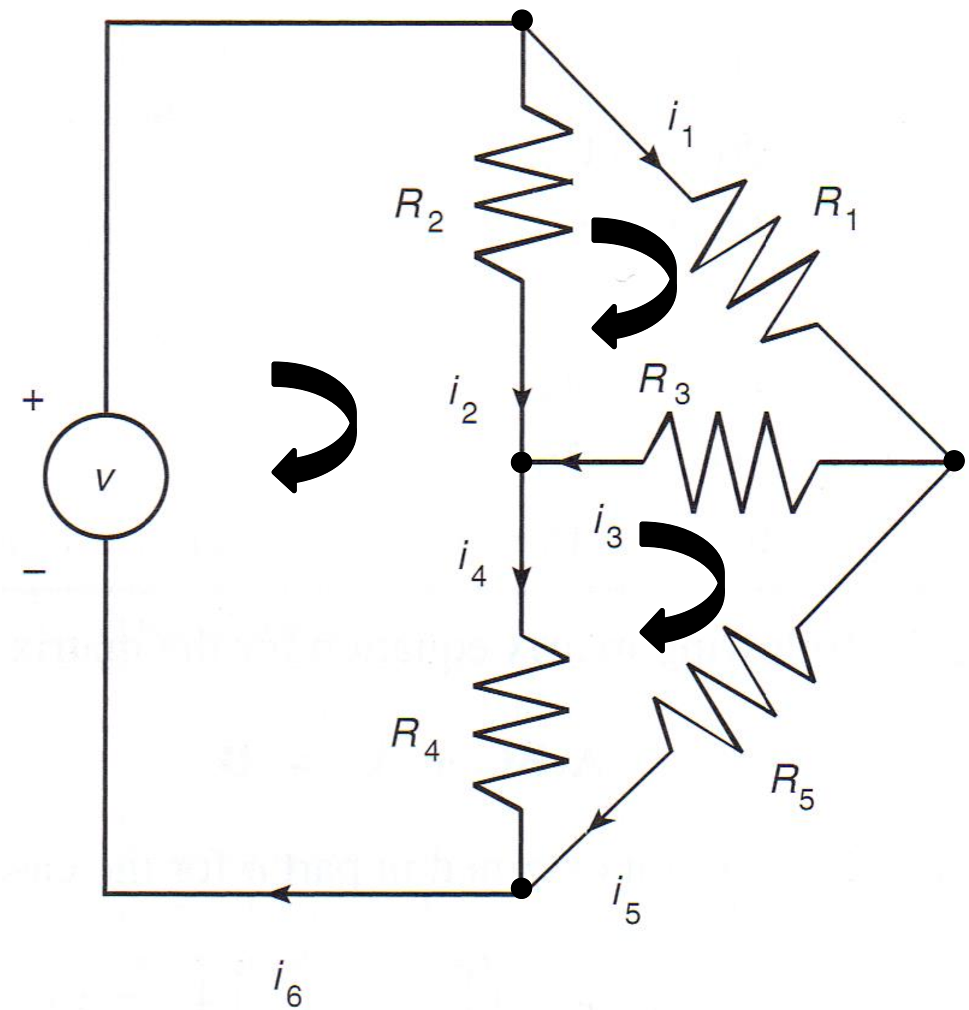
$$i_1 = i_3 + i_5$$

$$i_2 + i_3 = i_4$$

$$i_4 + i_5 = i_6$$

$$i_6 = i_1 + i_2$$

Unknowns: $i_1, i_2, i_3, i_4, i_5, i_6$



Problem 8.4:

$$(0)i_1 + (-R_2)i_2 + (0)i_3 + (-R_4)i_4 + (0)i_5 + (0)i_6 = (-v)$$

$$(-R_1)i_1 + (R_2)i_2 + (-R_3)i_3 + (0)i_4 + (0)i_5 + (0)i_6 = (0)$$

$$(0)i_1 + (0)i_2 + (R_3)i_3 + (R_4)i_4 + (-R_5)i_5 + (0)i_6 = (0)$$

$$(1)i_1 + (0)i_2 + (-1)i_3 + (0)i_4 + (-1)i_5 + (0)i_6 = (0)$$

$$(0)i_1 + (1)i_2 + (1)i_3 + (-1)i_4 + (0)i_5 + (0)i_6 = (0)$$

$$(0)i_1 + (0)i_2 + (0)i_3 + (1)i_4 + (1)i_5 + (-1)i_6 = (0)$$

$$(-1)i_1 + (-1)i_2 + (0)i_3 + (0)i_4 + (0)i_5 + (1)i_6 = (0)$$

Unknowns: $i_1, i_2, i_3, i_4, i_5, i_6$

Problem 8.4:

A =

0	-5000	0	-10000	0	0
-1000	5000	-2000	0	0	0
0	0	2000	10000	-5000	0
1	0	-1	0	-1	0
0	1	1	-1	0	0
0	0	0	1	1	-1
-1	-1	0	0	0	1

b =

-100
0
0
0
0
0
0

x =

0.0189
0.0049
0.0027
0.0076
0.0162
0.0238

Problem 8.11:

11.* Solve the following equations:

$$
\begin{aligned}
7x + 9y - 9z &= 22 \\
3x + 2y - 4z &= 12 \\
x + 5y - z &= -2
\end{aligned}
$$

Problem 8.11:

```
Command Window

Problem 8.11: Scott Thomas

A =

     7     9    -9
     3     2    -4
     1     5    -1


b =

    22
    12
    -2


RankA =

     2


RankAb =

     2
```

```
x1 =

    1.0000         0   -1.3846    4.9231
         0    1.0000    0.0769   -1.3846
         0         0         0         0


x2 =

    1.6437
   -1.2024
   -2.3684
```

The resulting equations are:

$$(1)x + (0)y - 1.3846z = 4.9231$$

$$(0)x + (1)y + 0.0769z = -1.3846$$

These equations can be solved in terms of z:

$$x = 1.3846z + 4.9231$$

$$y = -0.0769z - 1.3846$$

Problem 8.15:

15.* Use MATLAB to solve the following problem:

$$x - 3y = 2$$

$$x + 5y = 18$$

$$4x - 6y = 10$$

```
A =

     1    -3
     1     5
     4    -6


b =

     2
    18
    10
```

```
RankA =

     2


RankAb =

     3


x =

    6.0928
    2.2577
```