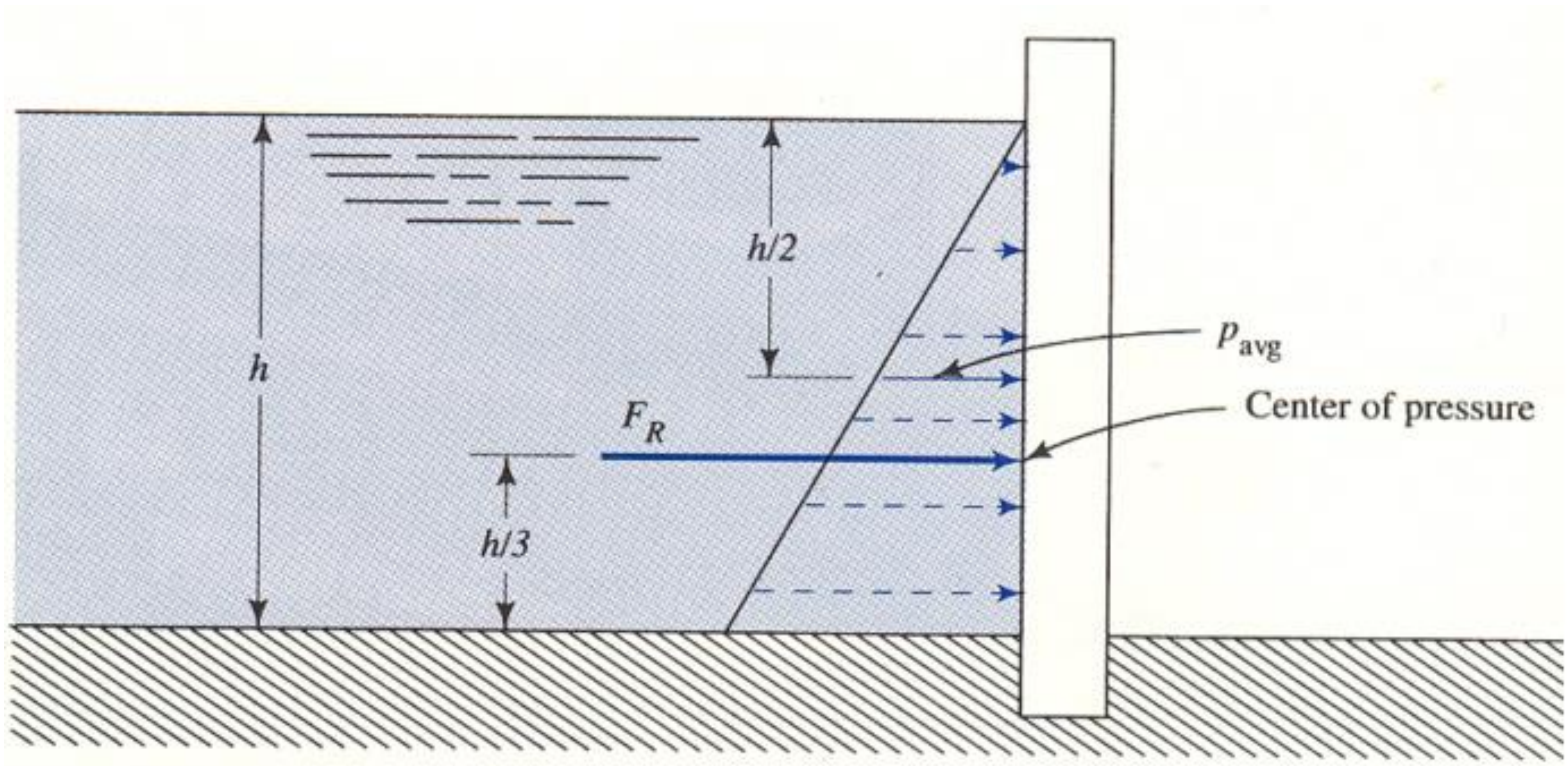
Chapter 9a: Numerical Methods for Calculus and Differential Equations

- Numerical Integration
- Numerical Differentiation

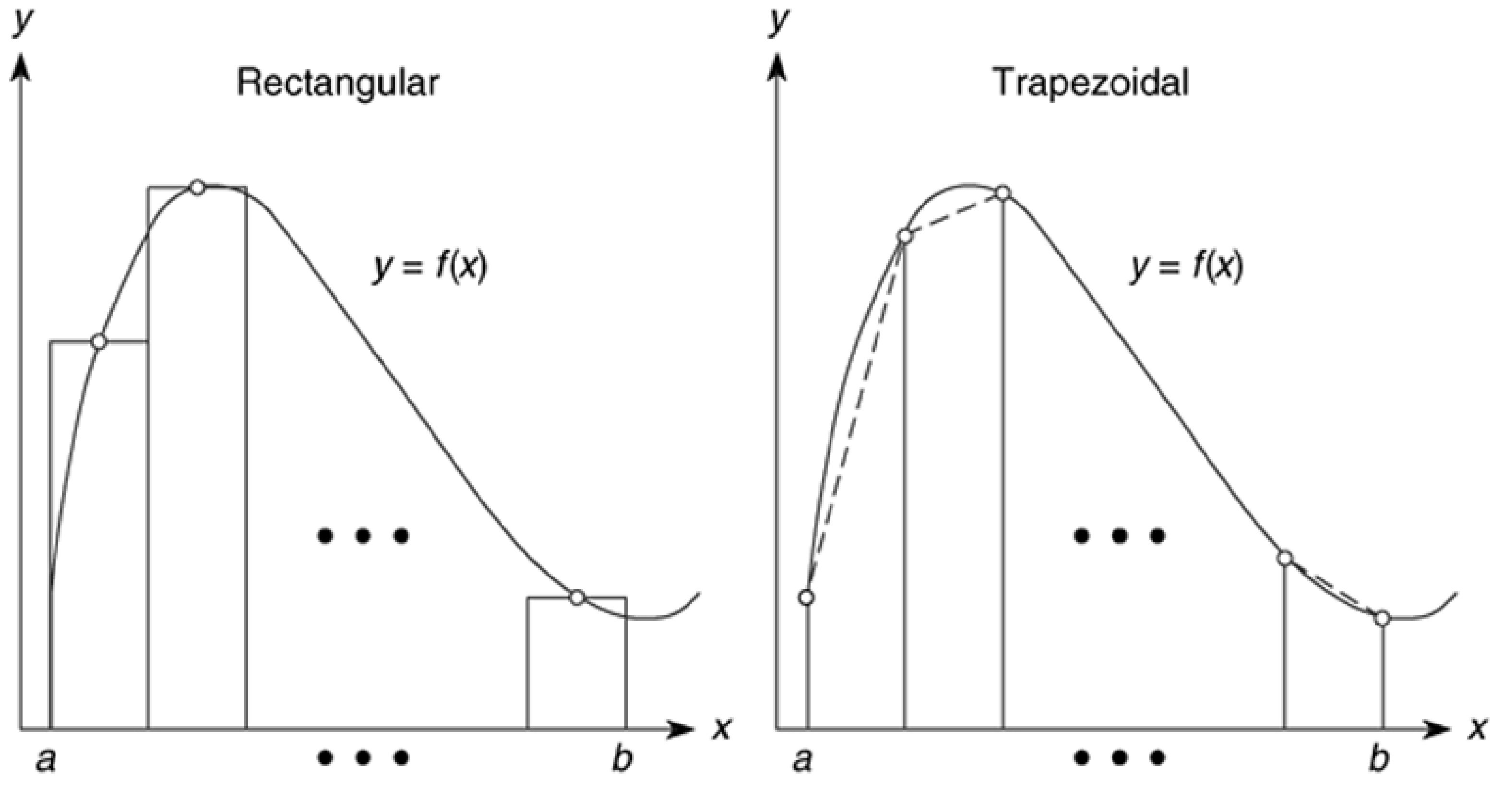
Integration

Integration is a very important mathematical concept that is used by engineers for many situations. For instance, the pressure distribution on a dam can be used to determine the center of pressure on the dam. The integral of the pressure distribution (the area under the curve) is the resultant force.



Numerical Integration

Numerical integration is used when the function can't be integrated directly. The area under the curve is estimated by dividing it using rectangular strips. A more accurate estimate is made by using trapezoidal strips.



(a) (b)

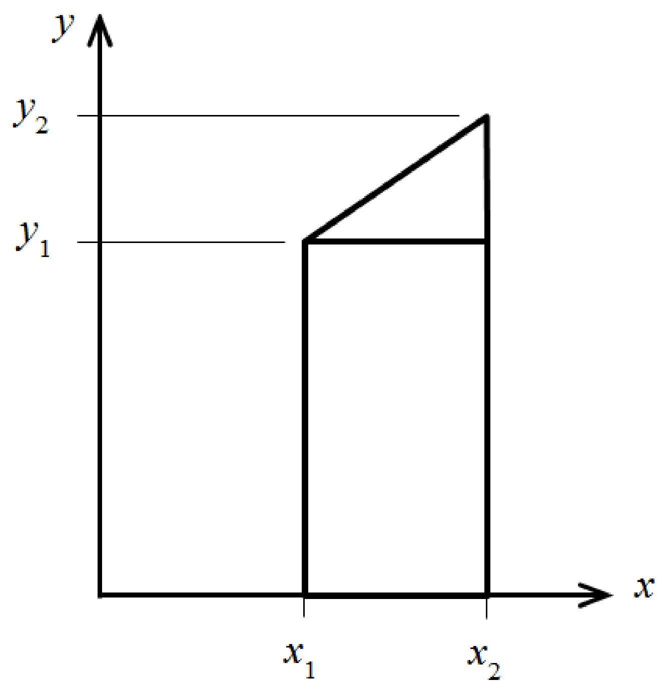
Numerical Integration

The area of a single trapezoid is given by:

$$A_{\text{trapezoid}} = (x_2 - x_1)y_1 + \frac{1}{2}(x_2 - x_1)(y_2 - y_1) = A_{\text{rectangle}} + A_{\text{triangle}}$$

$$A_{\text{trapezoid}} = (x_2 - x_1)\left(y_1 + \frac{1}{2}y_2 - \frac{1}{2}y_1\right)$$

$$A_{\text{trapezoid}} = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$$



Problem 9.3:

An object starts with an initial velocity of $v_0 = 3$ m/s at $t = 0$, and it accelerates at $a(t) = 7t\ \text{m/s}^2$. Find the total distance the object travels in $t = 4$ seconds.

Direct Integration of Functions

From Dynamics, the velocity as a function of time can be found by direct integration of the acceleration.

$$a(t) = \frac{dv}{dt} \quad \text{(acceleration as a function of time)}$$

$$dv = a\,dt \quad \text{(separate variables)}$$

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} a\,dt \quad \text{(integrate both sides)}$$

$$v - v_0 = \int_{t_0}^{t} a\,dt$$

Problem 9.3:

$$v(t) = v_0 + \int_{t_0}^{t} a\,dt \text{ (velocity as a function of time)}$$

Now find the distance traveled by direct integration of the velocity.

$$v(t) = \frac{dx}{dt} \text{ (velocity as a function of time)}$$

$$dx = v\,dt \text{ (separate variables)}$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v\,dt \text{ (integrate both sides)}$$

$$x - x_0 = \int_{t_0}^{t} v\,dt$$

$$x(t) = x_0 + \int_{t_0}^{t} v\,dt \text{ (distance traveled as a function of time)}$$

Problem 9.3:

For this problem, the acceleration, initial velocity and initial position are:

$$a(t) = 7t \text{ m/s}^2, \qquad v(t=0) = v_0 = 3.0 \text{ m/s},$$
$$x(t=0) = x_0 = 0.0 \text{ m}$$

Integrate the acceleration to determine the velocity:

$$v(t) = v_0 + \int_{t_0}^{t} a\, dt = 3.0 + \int_{0}^{t} (7t)\, dt$$

$$v(t) = 3.0 + \left[7\left(\frac{t^2}{2}\right)\right]_0^t$$

$$v(t) = 3.0 + \left[7\left(\frac{t^2}{2}\right) - 7\left(\frac{0^2}{2}\right)\right]$$

$$v(t) = 3.0 + \left(\frac{7}{2}\right)t^2$$

$$v(t = 4 \text{ seconds}) = 3.0 + \left(\frac{7}{2}\right)(4)^2 = 59.0 \text{ m/s}$$

Problem 9.3:

Now that we have the velocity as a function of time $v(t)$, we can integrate to find the distance traveled $x(t)$. The initial position is $x(t = 0) = 0.0$ m:

$$x(t) = x_0 + \int_{t_0}^{t} v\,dt = 0.0 + \int_0^t \left[3 + \left(\frac{7}{2}\right) t^2\right] dt$$

$$x(t) = \left[3t + \frac{7}{2}\left(\frac{t^3}{3}\right)\right]_0^t$$

$$x(t) = \left[3t + \frac{7}{2}\left(\frac{t^3}{3}\right)\right] - \left[3(0) + \frac{7}{2}\left(\frac{(0)^3}{3}\right)\right]$$

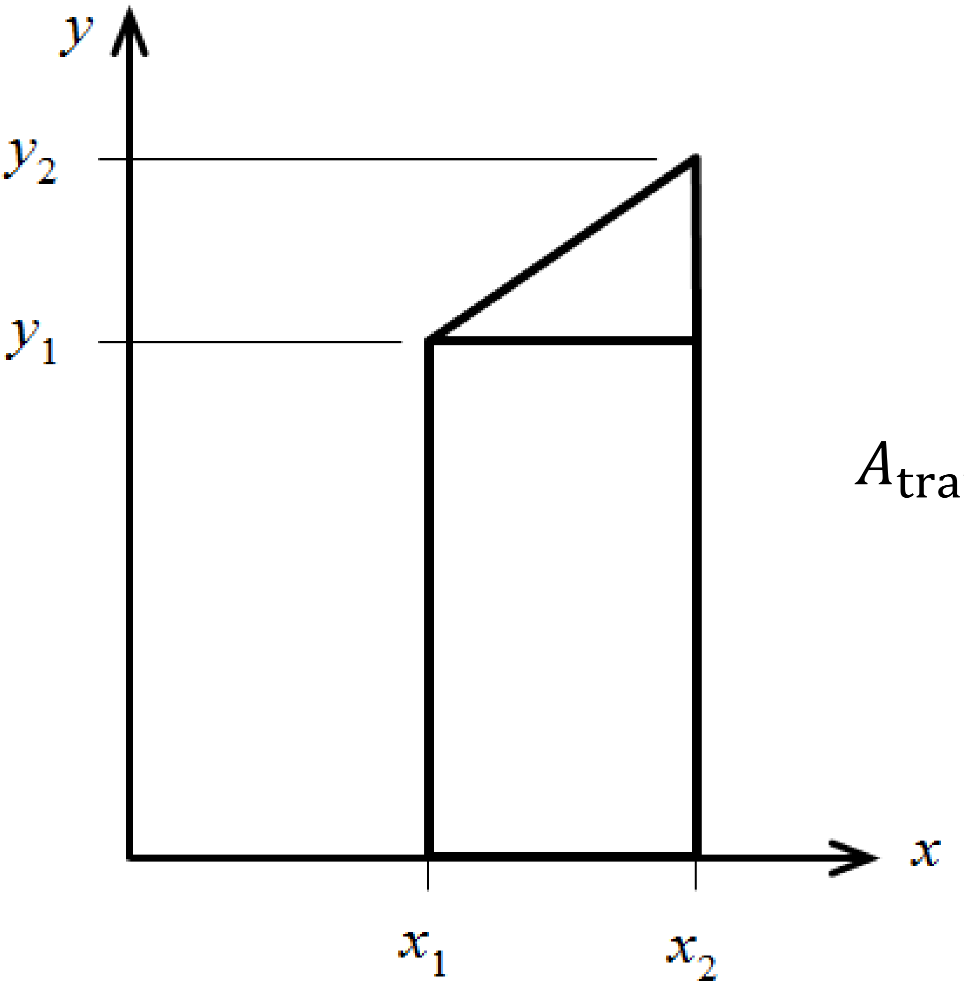
$$x(t) = 3t + \frac{7}{2}\left(\frac{t^3}{3}\right)$$

$$x(t = 4\text{ seconds}) = 3(4) + \frac{7}{2}\left[\frac{(4)^3}{3}\right] = 86.\bar{6}\text{ m}$$

Problem 9.3:

Numerical Integration using the Trapezoidal Rule

If the direct integration method cannot be used, the function can be numerically integrated by approximating the area under the curve using trapezoids in a piecewise manner. Recall the area of a trapezoid:



$$A_{\text{trapezoid}} = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$$

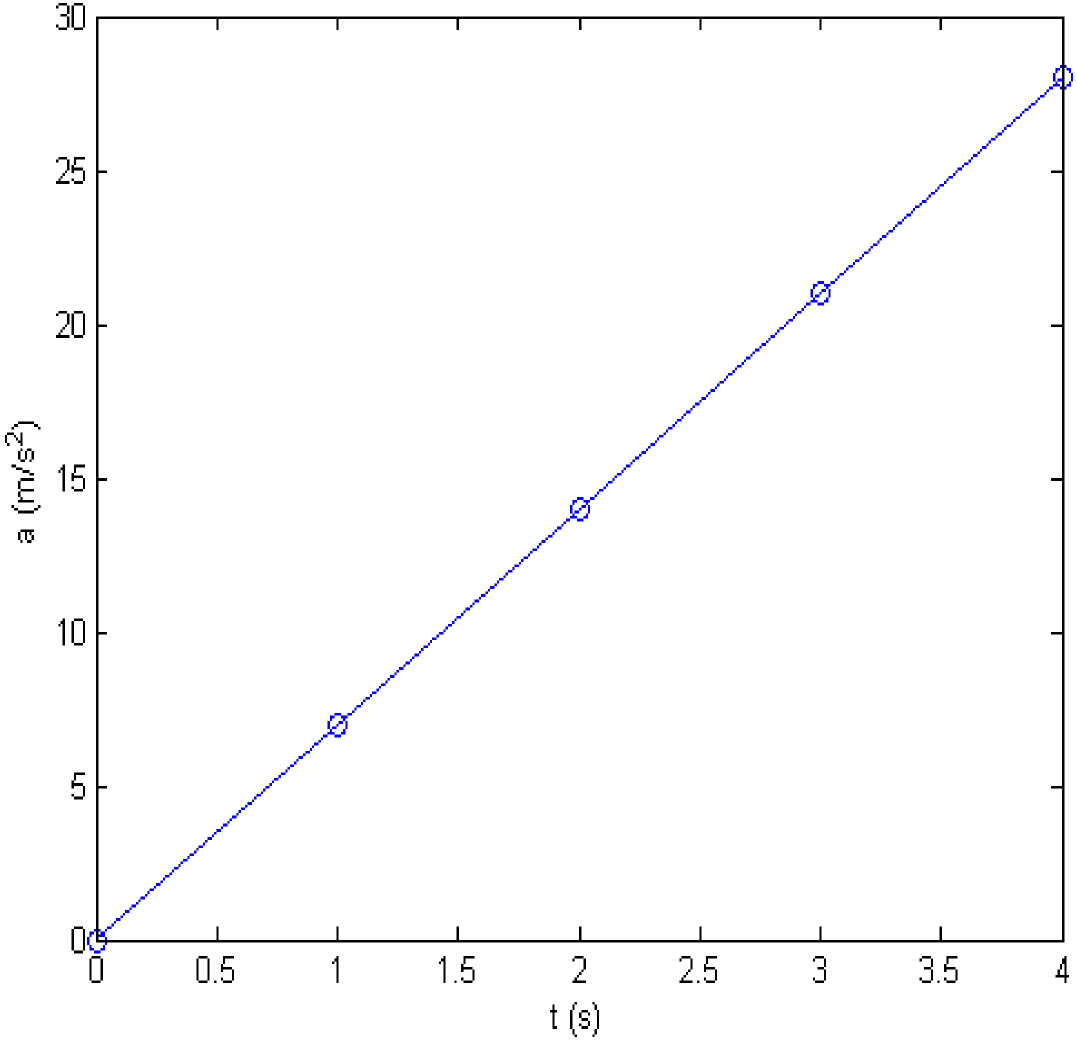
Problem 9.3:
Numerical Integration using the Trapezoidal Rule

For Problem 9.3, divide the time $0 \leq t \leq 4.0$ seconds into four equal periods by letting $N = 5$. Evaluate the acceleration function at each point in time:

$$a(t) = 7t \ \frac{\mathrm{m}}{\mathrm{s}^2}$$

```
N = 5;
t = linspace(0,4,N);
a = 7*t;
plot(t,a,'-o'),xlabel('t (s)'),ylabel('a (m/s^2)')
```

Problem 9.3:



$$a(t) = 7t \ \frac{\text{m}}{\text{s}^2}$$

$$a_1 = 7t_1 = 7(0.0) = 0.0$$
$$a_2 = 7t_2 = 7(1.0) = 7.0$$
$$a_3 = 7t_3 = 7(2.0) = 14.0$$
$$a_4 = 7t_4 = 7(3.0) = 21.0$$
$$a_5 = 7t_5 = 7(4.0) = 28.0$$

In general, $a_k = 7t_k$, where k will be the **for loop** counter.

Problem 9.3:

Numerically integrate the acceleration $a(t)$ to find the velocity. The initial velocity is $v(t = 0) = 3.0$ m/s. Remember that the first index for a vector in MATLAB is $k = 1$. The general form for numerically integrating the acceleration using the **Trapezoidal Rule** is:

$$v_{k+1} = v_k + \frac{1}{2}(t_{k+1} - t_k)(a_k + a_{k+1})$$

$$v_1 = 3.0$$

$$v_2 = v_1 + \frac{1}{2}(t_2 - t_1)(a_1 + a_2) = 3.0 + \frac{1}{2}(1.0 - 0.0)(0.0 + 7.0) = 6.5$$

$$v_3 = v_2 + \frac{1}{2}(t_3 - t_2)(a_2 + a_3) = 6.5 + \frac{1}{2}(2.0 - 1.0)(7.0 + 14.0) = 17.0$$

$$v_4 = v_3 + \frac{1}{2}(t_4 - t_3)(a_3 + a_4) = 17.0 + \frac{1}{2}(3.0 - 2.0)(14.0 + 21.0) = 34.5$$

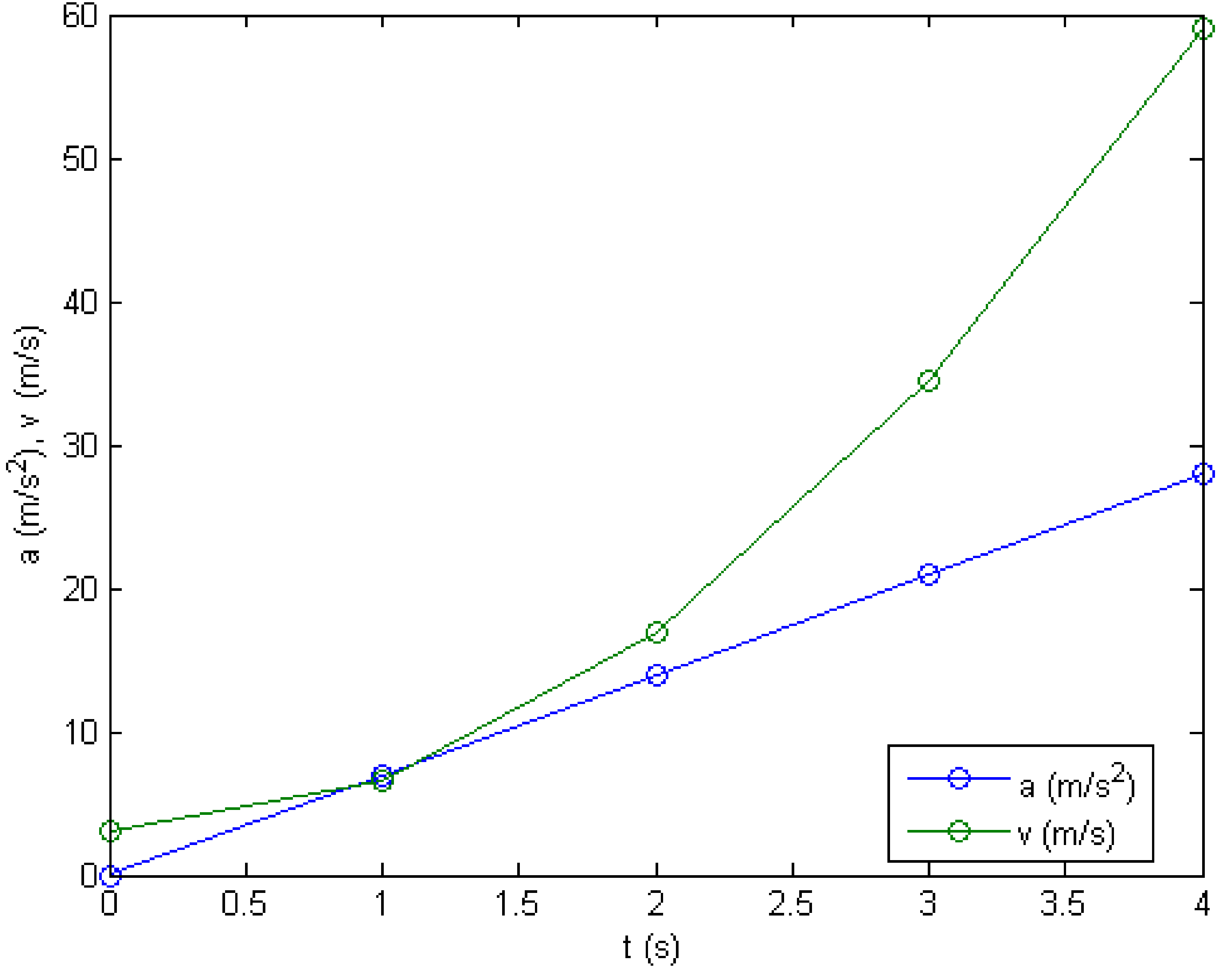
$$v_5 = v_4 + \frac{1}{2}(t_5 - t_4)(a_4 + a_5) = 34.5 + \frac{1}{2}(4.0 - 3.0)(21.0 + 28.0) = 59.0$$

Problem 9.3:

The initial velocity is $v(t = 0) = 3.0$ m/s. The **for loop** loads the velocity vector during the integration process. Use the **Debugging Tool** to see the values. Check the velocity values to make sure they are the same was what we calculated by hand.

```
N = 5;
t = linspace(0,4,N);
a = 7*t;

v(1) = 3.0;
for k = 1:N-1
    v(k+1) = v(k) + 0.5*(t(k+1) - t(k))*(a(k) + a(k+1));
end

plot(t,a,'-o',t,v,'-o'),xlabel('t (s)'),ylabel('a (m/s^2), v (m/s)')
legend('a (m/s^2)', 'v (m/s)', 'Location', 'Best')
```

Problem 9.3:



Problem 9.3:

Numerically integrate the velocity $v(t)$ to find the distance traveled. The initial position is $x(t = 0) = 0.0$ m. The general form for numerically integrating the velocity is:

$$x_{k+1} = x_k + \frac{1}{2}(t_{k+1} - t_k)(v_k + v_{k+1})$$

$$x_1 = 0.0$$

$$x_2 = x_1 + \frac{1}{2}(t_2 - t_1)(v_1 + v_2) = 0.0 + \frac{1}{2}(1.0 - 0.0)(3.0 + 6.5) = 4.75$$

$$x_3 = x_2 + \frac{1}{2}(t_3 - t_2)(v_2 + v_3) = 4.75 + \frac{1}{2}(2.0 - 1.0)(6.5 + 17.0) = 16.5$$

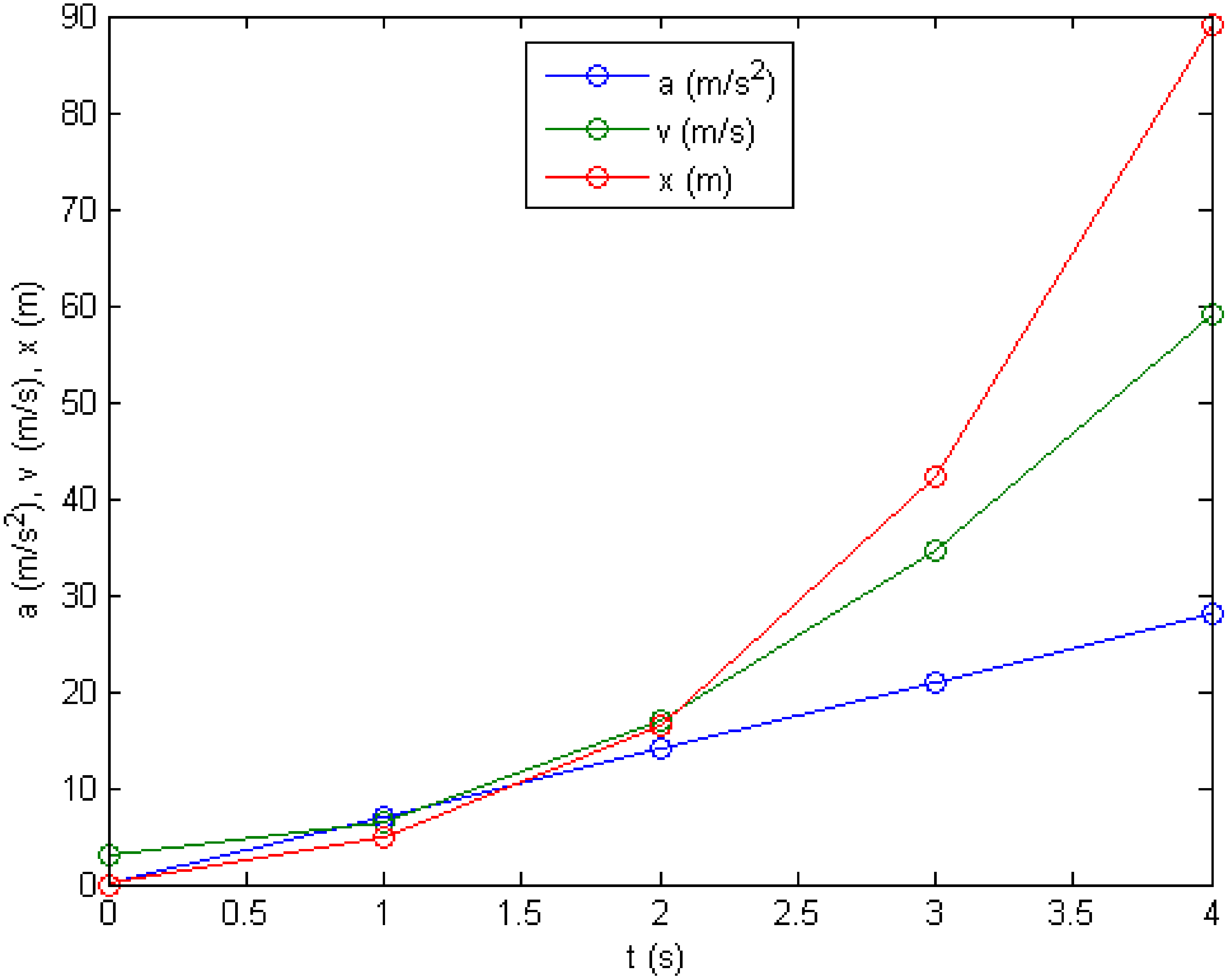
$$x_4 = x_3 + \frac{1}{2}(t_4 - t_3)(v_3 + v_4) = 16.5 + \frac{1}{2}(3.0 - 2.0)(17.0 + 34.5) = 42.25$$

$$x_5 = x_4 + \frac{1}{2}(t_5 - t_4)(v_4 + v_5) = 42.25 + \frac{1}{2}(4.0 - 3.0)(34.5 + 59.0) = 89.0$$

Problem 9.3:

```
N = 5;
t = linspace(0,4,N);
a = 7*t;

v(1) = 3.0;
for k = 1:N-1
    v(k+1) = v(k) + 0.5*(t(k+1) - t(k))*(a(k) + a(k+1));
end

fprintf('v(N)= %4.2f m/s \n',v(N))

x(1) = 0;
for k = 1:N-1
    x(k+1) = x(k) + 0.5*(t(k+1) - t(k))*(v(k) + v(k+1));
end

fprintf('x(N)= %4.2f m \n',x(N))

plot(t,a,'-o', t,v,'-o',t,x,'-o')
xlabel('t (s)')
ylabel('a (m/s^2), v (m/s), x (m)')
legend('a (m/s^2)', 'v (m/s)', 'x (m)', 'Location', 'Best')
```

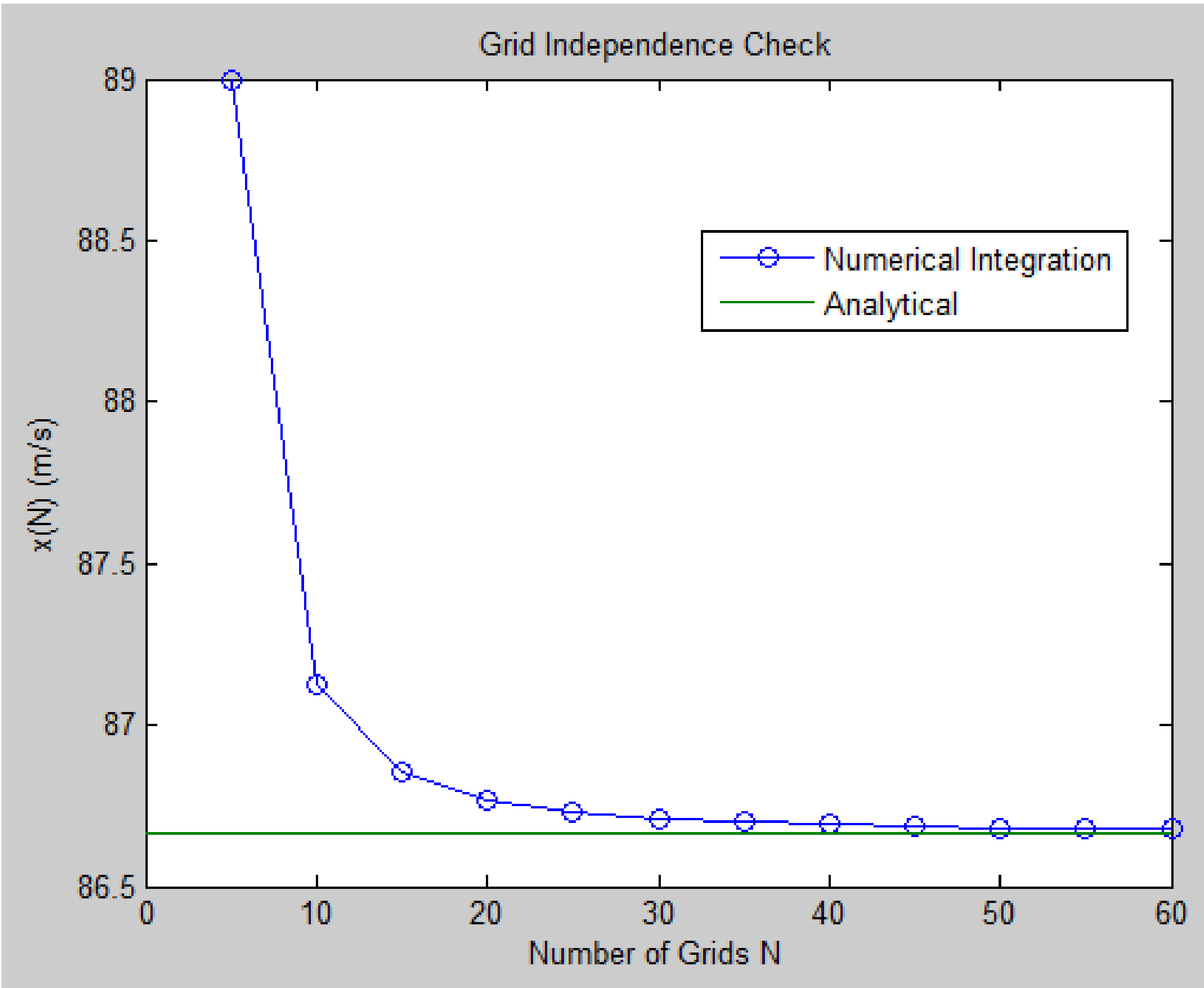
Problem 9.3:



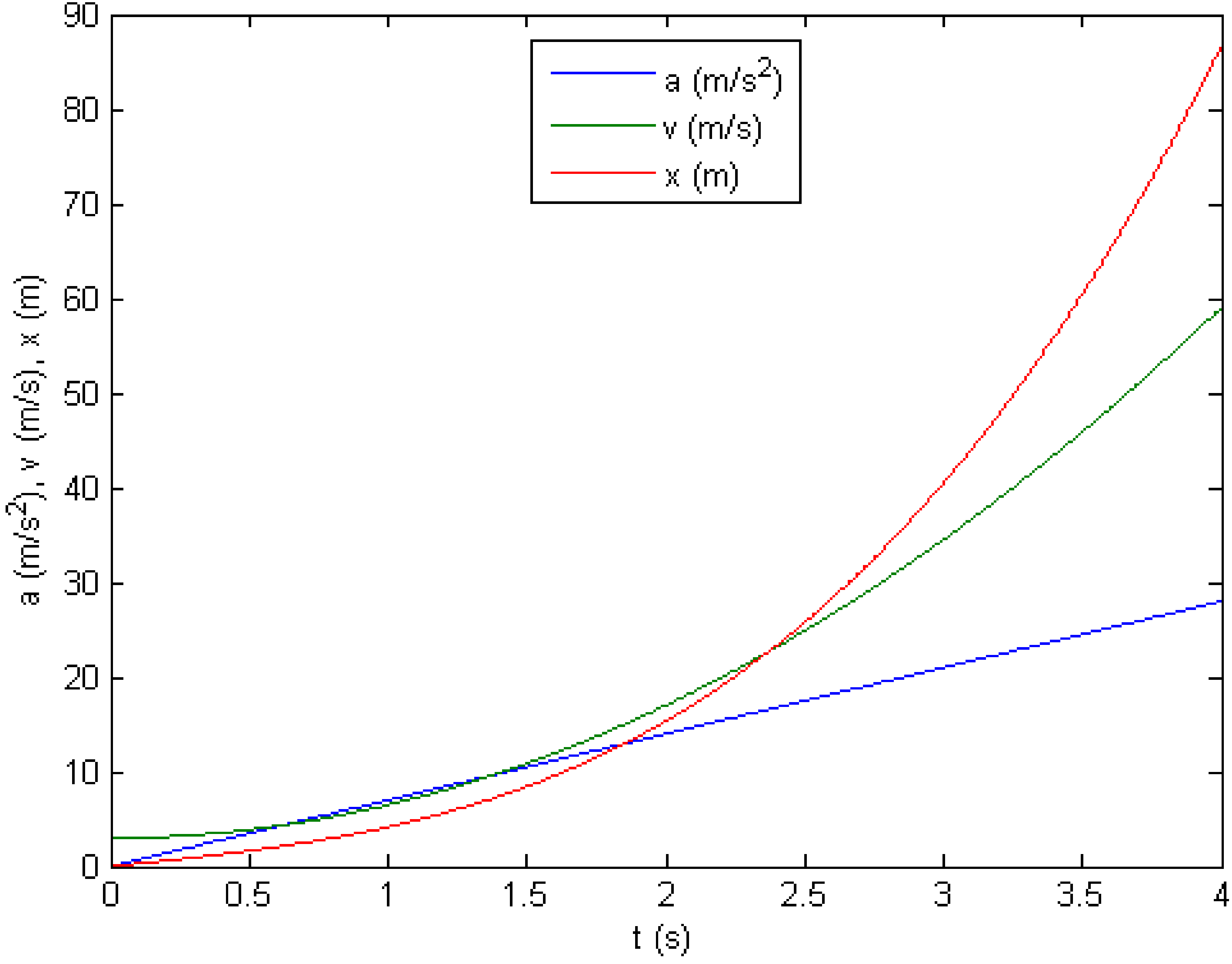
a (m/s^2)
v (m/s)
x (m)
a (m/s^2), v (m/s), x (m)
t (s)
0
0.5
1
1.5
2
2.5
3
3.5
4
0
10
20
30
40
50
60
70
80
90

Problem 9.3:

For $N = 5$, the distance traveled is predicted to be $x_N = 89.0$ m, which does not match the analytical solution ($x = 86.\bar{6}$ m). Improve the numerical integration prediction by increasing N:



Problem 9.3:



90
80
70
60
50
40
30
20
10
0
0
0.5
1
1.5
2
2.5
3
3.5
4
t (s)
a (m/s2), v (m/s), x (m)
a (m/s2)
v (m/s)
x (m)

Problem 9.1:

An object moves at a velocity $v(t) = 5 + 7t^2$ m/s starting from the position $x(2) = 5$ m at $t_0 = 2$ seconds. Determine its position at $t = 10$ seconds.

$$x(t) = x_0 + \int_{t_0}^{t} v\,dt$$

$$x(t) = x_0 + \int_{t_0}^{t} (5 + 7t^2)dt$$

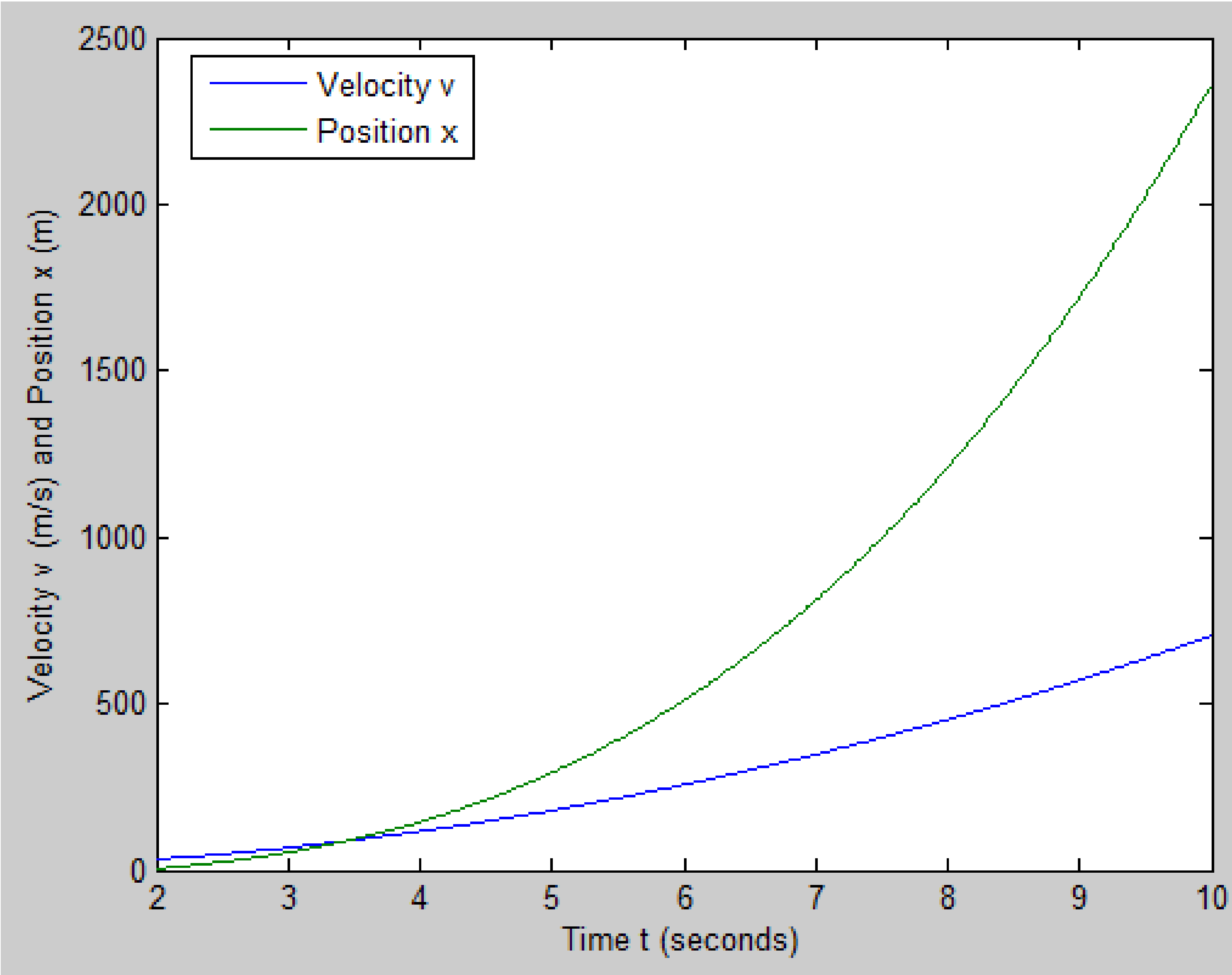
$$x(t) = x_0 + \left[5t + \frac{7}{3}t^3\right]_{t_0}^{t}$$

$$x(t) = x_0 + 5(t - t_0) + \frac{7}{3}(t^3 - t_0^3)$$

$$x(10) = 5 + 5(10 - 2) + \frac{7}{3}(10^3 - 2^3) = 2359.\overline{6}$$

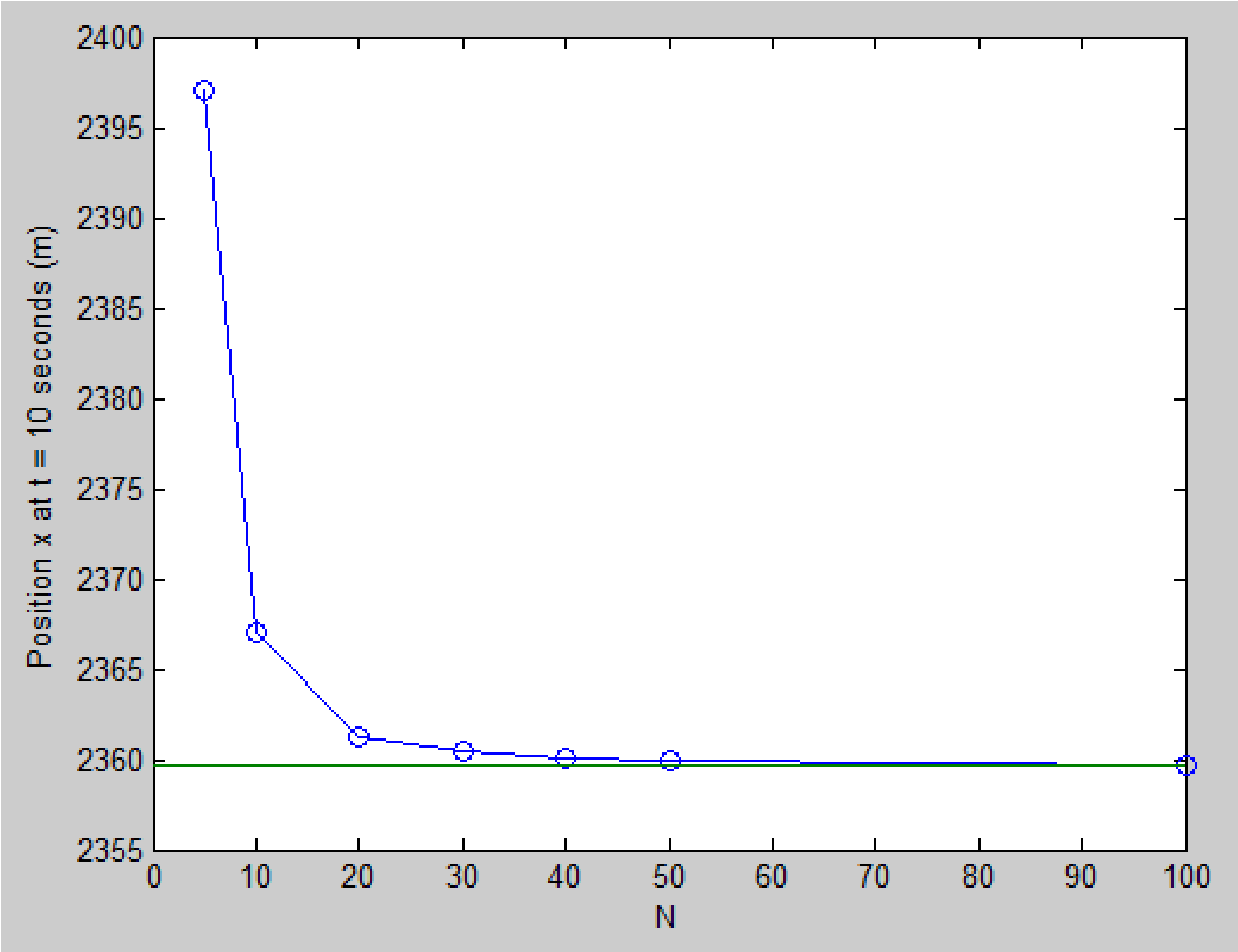
$$x_{k+1} = x_k + \frac{1}{2}(t_{k+1} - t_k)(v_k + v_{k+1})$$

Problem 9.1:



Velocity v
Position x
Velocity v (m/s) and Position x (m)
Time t (seconds)
0
500
1000
1500
2000
2500
2
3
4
5
6
7
8
9
10

Problem 9.1:



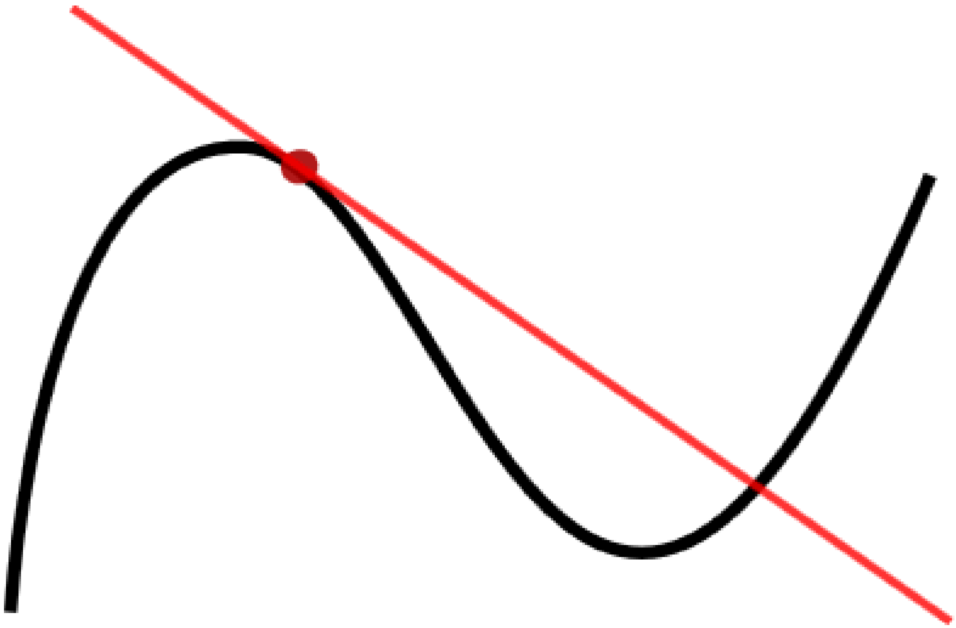
Position x at t = 10 seconds (m)
N
2400
2395
2390
2385
2380
2375
2370
2365
2360
2355
0
10
20
30
40
50
60
70
80
90
100

Differentiation

Differentiation of a function is the act of calculating the derivative of the function at any point.

The derivative is the slope of the curve, which is the tangent line shown below as a red line.

The inverse of differentiation is integration.



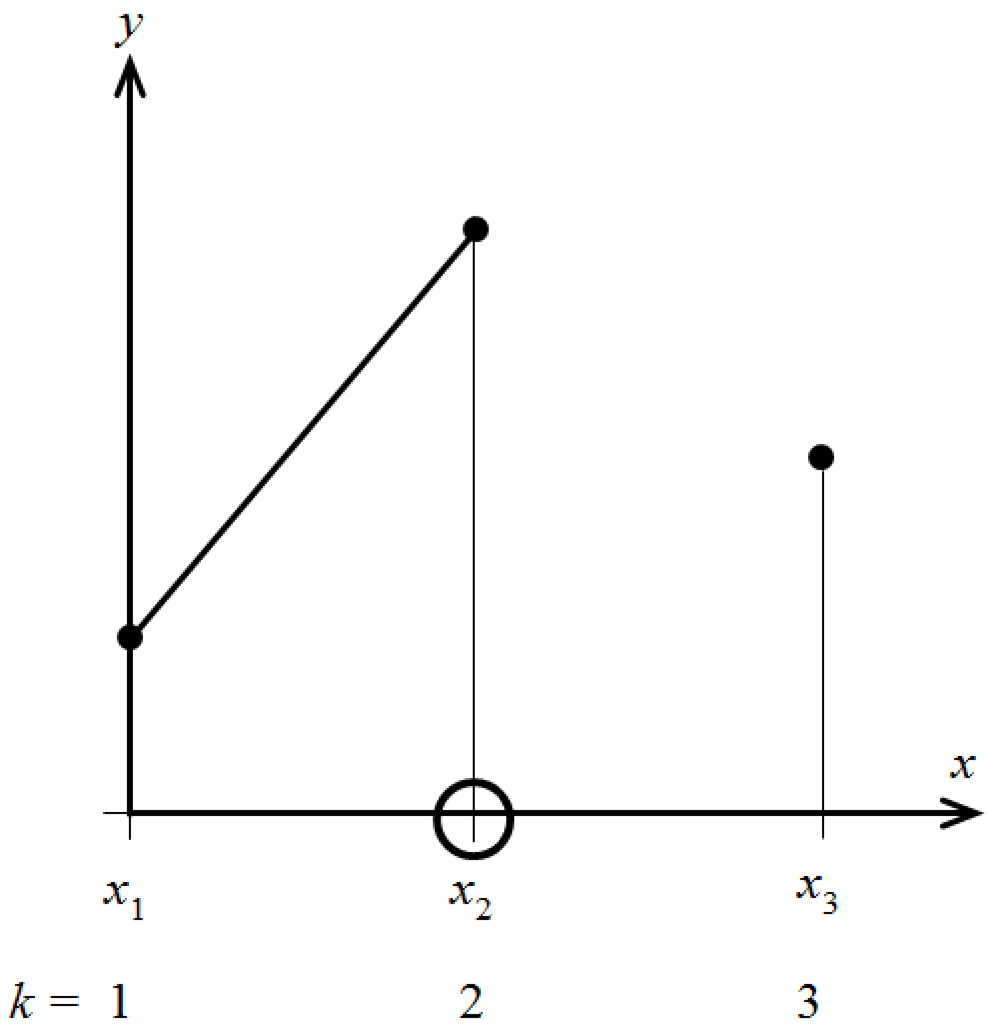
Numerical Differentiation

Numerical differentiation is used for finding the slope of functions that are given by discrete data points, such as experimental data. Three methods are used:

- Backward Difference
- Forward Difference
- Central Difference

Backward Difference

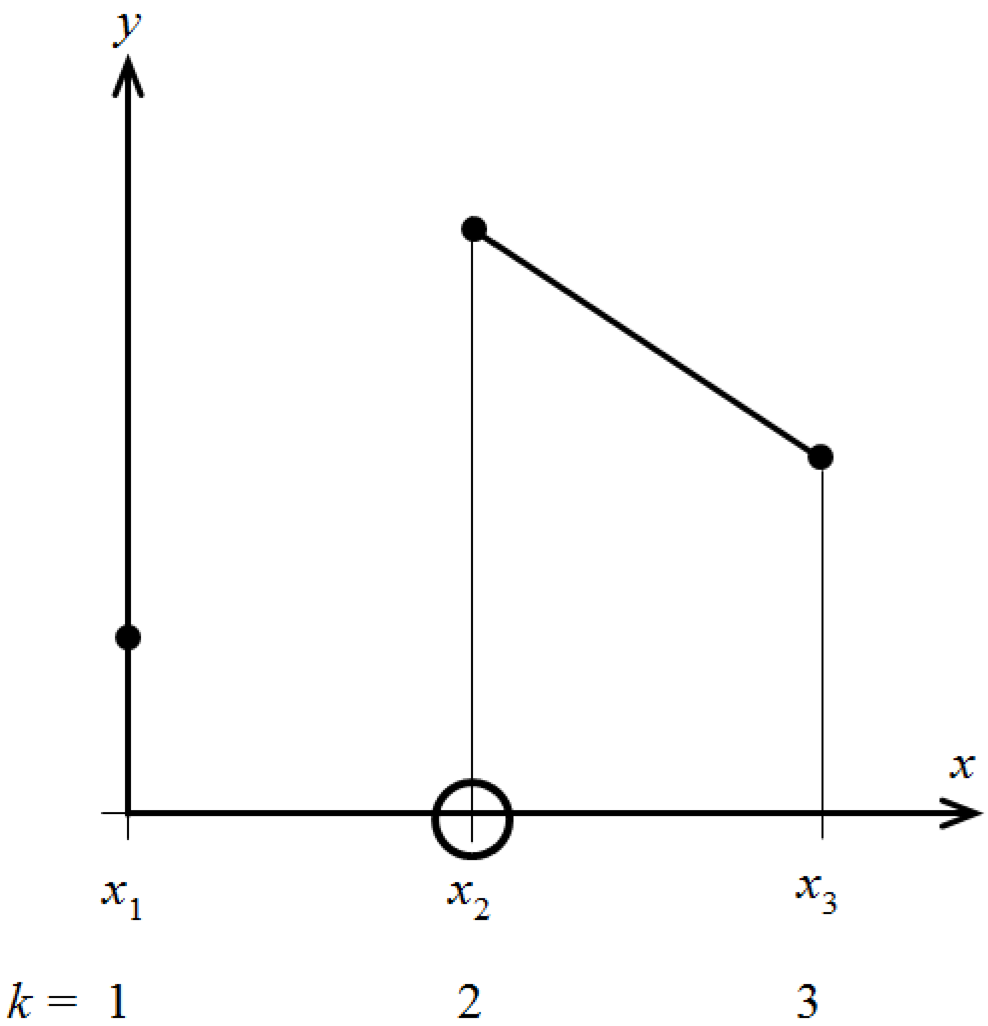
Estimate the derivative or slope at a point (dy/dx) by looking at the data point to the left of the point of interest.



$$\left.\frac{dy}{dx}\right|_{B2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Forward Difference

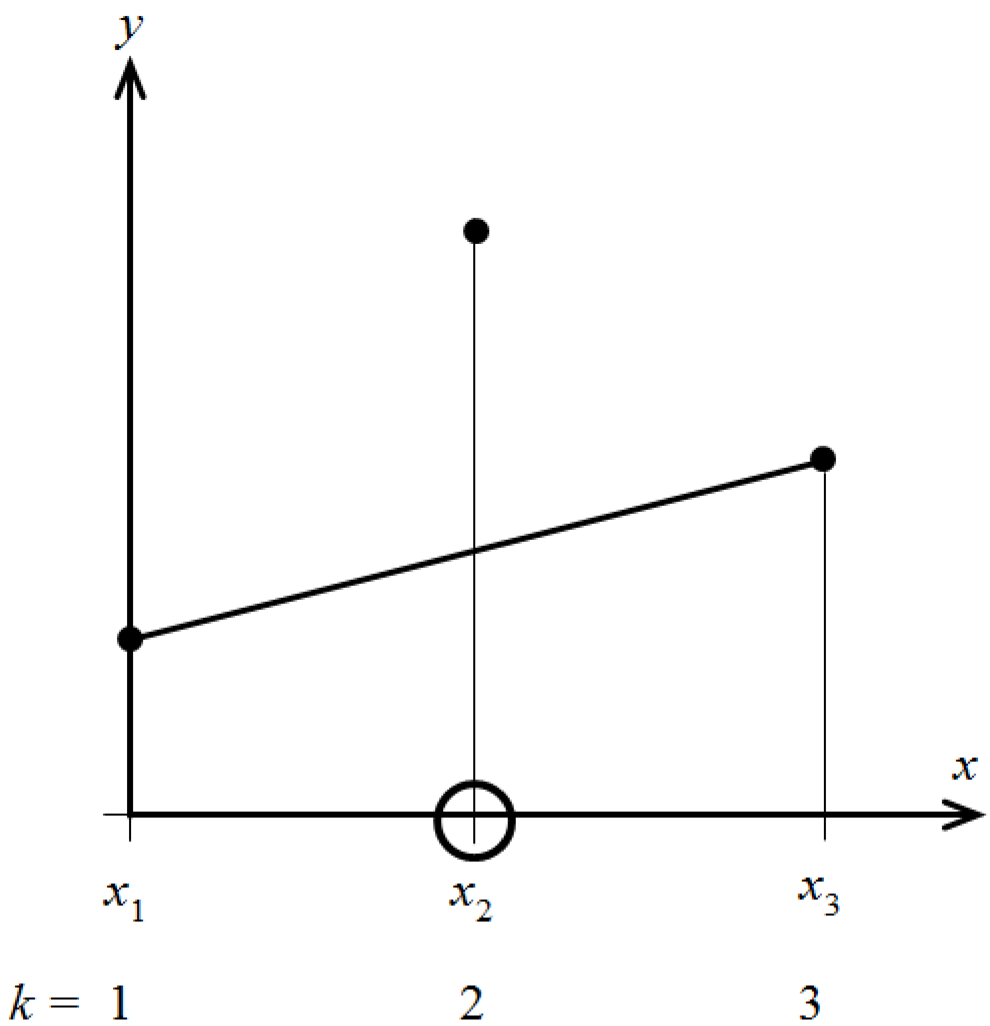
Estimate the derivative or slope at a point (dy/dx) by looking at the data point to the right of the point of interest.



$$\left.\frac{dy}{dx}\right|_{F2} = \frac{y_3 - y_2}{x_3 - x_2}$$

Central Difference

Estimate the derivative or slope at a point (dy/dx) by looking at the data points to the left and to the right of the point of interest.



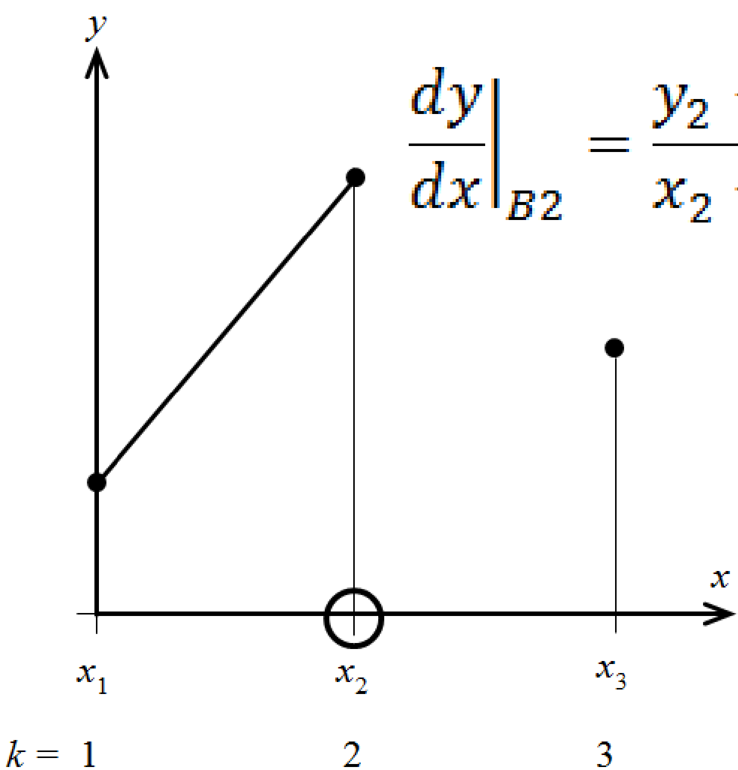
$$\left.\frac{dy}{dx}\right|_{C2} = \frac{y_3 - y_1}{x_3 - x_1}$$

Problem 9.17:

Plot the estimate of the derivative dy/dx from the following data. Do this by using forward, backward and central differences. Compare the results.

k	1	2	3	4	5	6	7	8	9	10	$N = 11$
x	0	1	2	3	4	5	6	7	8	9	10
y	0	2	5	7	9	12	15	18	22	20	17

Backward Difference: Estimate dy/dx by looking backward.



$$\left.\frac{dy}{dx}\right|_{B2} = \frac{y_2 - y_1}{x_2 - x_1}$$

In general, the backward difference is given in terms of the **for loop** counter k:

$$\left.\frac{dy}{dx}\right|_{B,k} = \frac{y_k - y_{k-1}}{x_k - x_{k-1}}$$

$$\left.\frac{dy}{dx}\right|_{B2} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{1 - 0} = 2$$

Problem 9.17:

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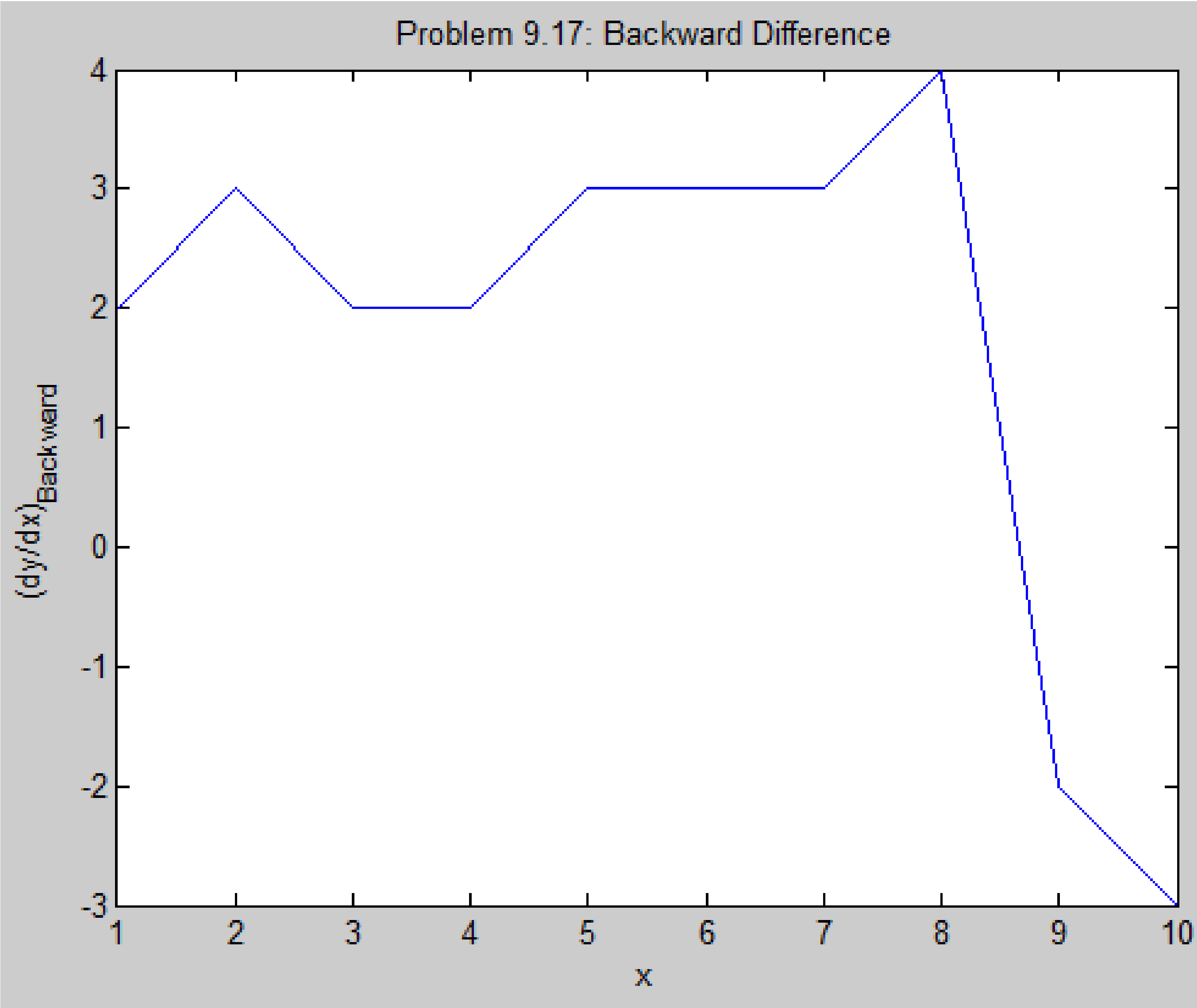
problem9_17.m

```matlab
clc
clear
N = 11;
x = linspace(0,10,N);
y = [0   2    5    7    9    12   15   18   22   20   17];
% Backward difference
for k=2:N
    dydx_b(k) = (y(k) - y(k-1))/(x(k) - x(k-1));
end
plot(x(2:N),dydx_b(2:N))
xlabel('x'), ylabel('(dy/dx)_{Backward}')
title('Problem 9.17: Backward Difference')
```

Problem 9.17:

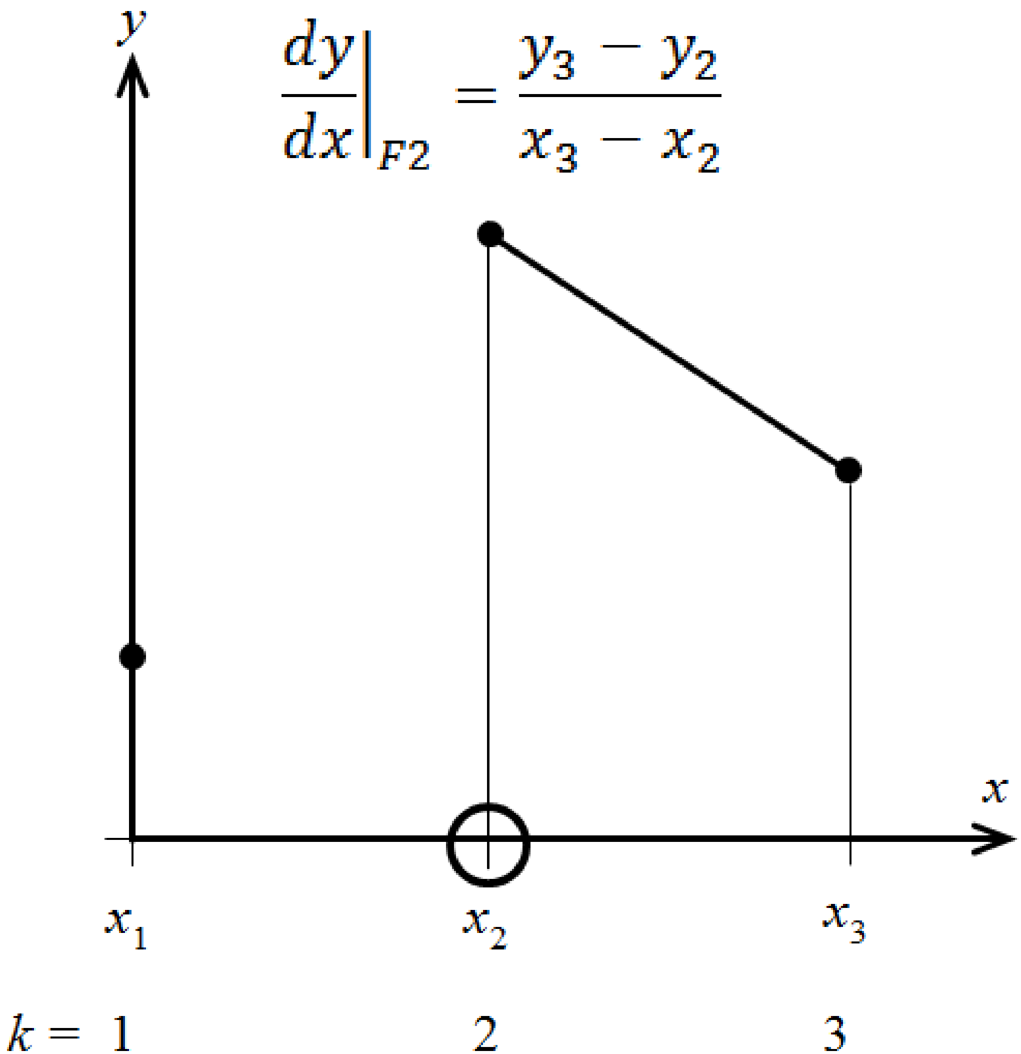


Problem 9.17: Backward Difference
4
3
2
1
0
-1
-2
-3
1
2
3
4
5
6
7
8
9
10
x
$(dy/dx)_{Backward}$

Problem 9.17:

Forward Difference: Estimate dy/dx by looking forward.

$$\left.\frac{dy}{dx}\right|_{F2} = \frac{y_3 - y_2}{x_3 - x_2}$$



In general, the forward difference is given in terms of the **for loop** counter k:

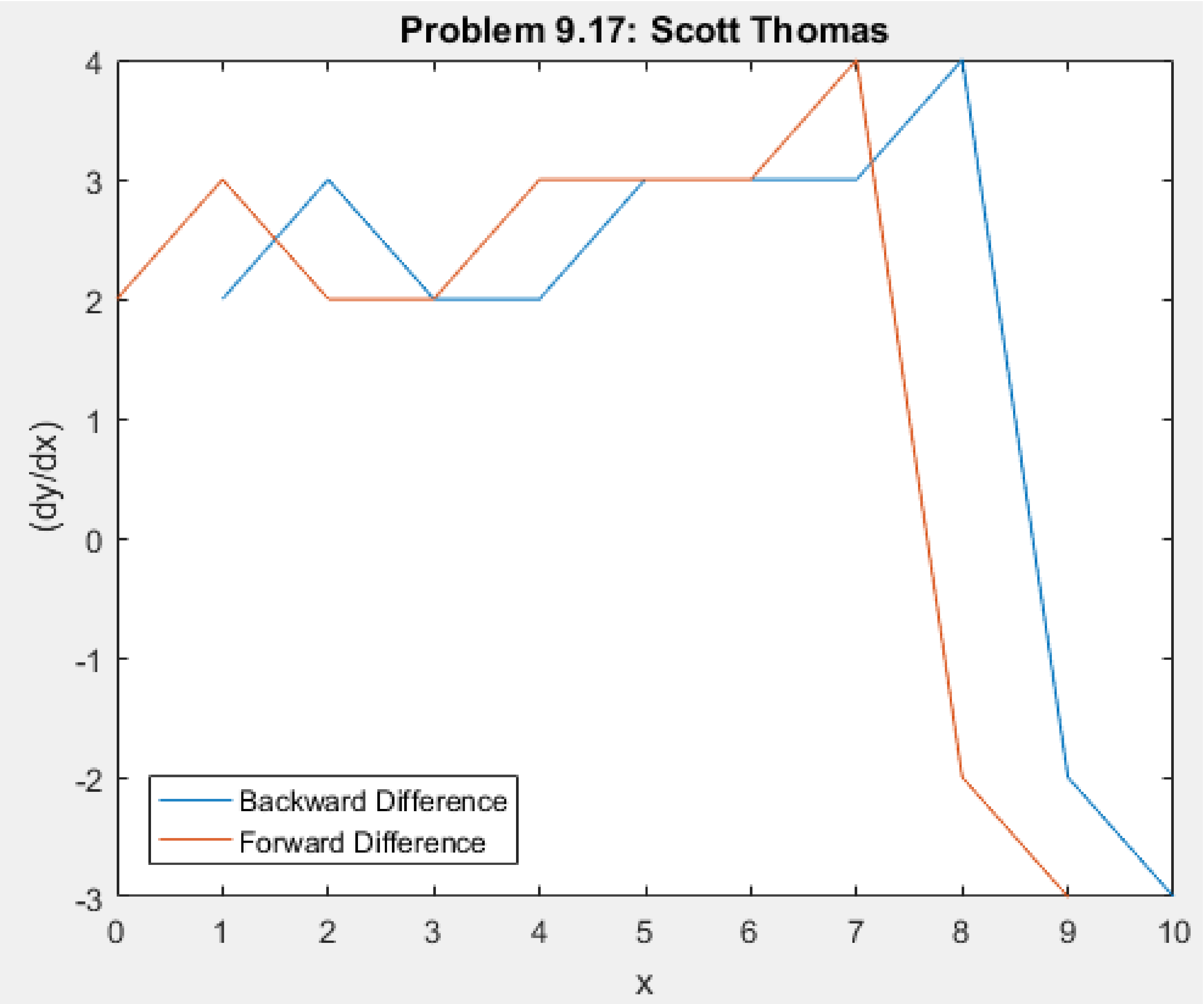
$$\left.\frac{dy}{dx}\right|_{F,k} = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$$

$$\left.\frac{dy}{dx}\right|_{F1} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{1 - 0} = 2$$

Problem 9.17:

```
clc
clear
N = 11;
x = linspace(0,10,N);
y = [0   2    5    7    9    12   15   18   22   20   17];
% Backward difference
for k=2:N
    dydx_b(k) = (y(k) - y(k-1))/(x(k) - x(k-1));
end
% Forward difference
for k=1:N-1
    dydx_f(k) = (y(k+1) - y(k))/(x(k+1) - x(k));
end
plot(x(2:N),dydx_b(2:N),x(1:N-1),dydx_f(1:N-1))
xlabel('x'), ylabel('(dy/dx)')
title('Problem 9.17: Scott Thomas')
legend('Backward Difference','Forward Difference', 'Location','SouthWest'
```

Problem 9.17:

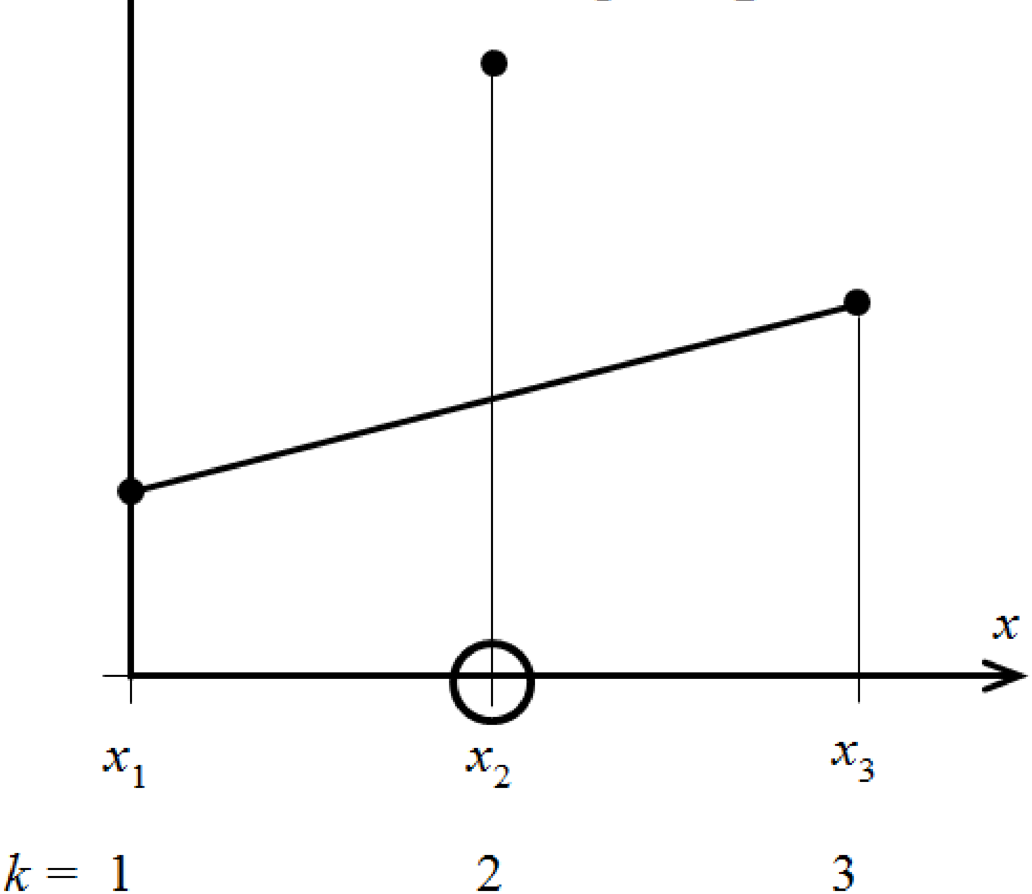


Problem 9.17: Scott Thomas
(dy/dx)
x
Backward Difference
Forward Difference
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6
7
8
9
10

<u>**Problem 9.17:**</u>

Central Difference: Estimate dy/dx by looking both backward and forward.

$$\left.\frac{dy}{dx}\right|_{C2} = \frac{y_3 - y_1}{x_3 - x_1}$$



In general, the forward difference is given in terms of the **for loop** counter k:

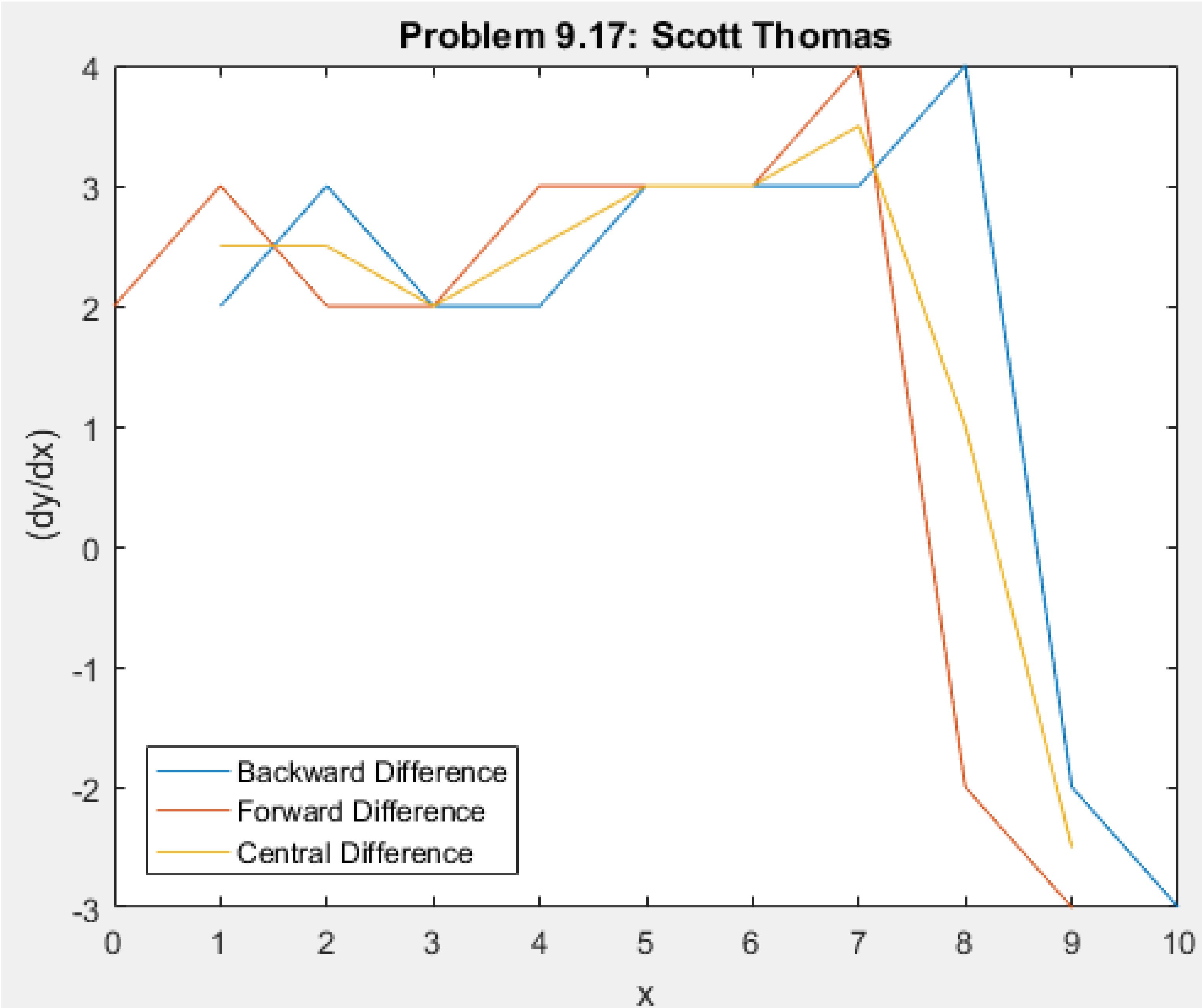
$$\left.\frac{dy}{dx}\right|_{C,k} = \frac{y_{k+1} - y_{k-1}}{x_{k+1} - x_{k-1}}$$

$$\left.\frac{dy}{dx}\right|_{C2} = \frac{y_3 - y_1}{x_3 - x_1} = \frac{5-0}{2-0} = \frac{5}{2}$$

Problem 9.17:

```
clc
clear
N = 11;
x = linspace(0,10,N);
y = [0  2   5   7   9   12  15  18  22  20  17];
% Backward difference
for k=2:N
    dydx_b(k) = (y(k) - y(k-1))/(x(k) - x(k-1));
end
% Forward difference
for k=1:N-1
    dydx_f(k) = (y(k+1) - y(k))/(x(k+1) - x(k));
end
% Central difference
for k=2:N-1
    dydx_c(k) = (y(k+1) - y(k-1))/(x(k+1) - x(k-1));
end
plot(x(2:N),dydx_b(2:N),x(1:N-1),dydx_f(1:N-1),x(2:N-1),dydx_c(2:N-1))
xlabel('x'), ylabel('(dy/dx)')
title('Problem 9.17: Scott Thomas')
legend('Backward Difference','Forward Difference',...
    'Central Difference', 'Location','SouthWest')
```

Problem 9.17:

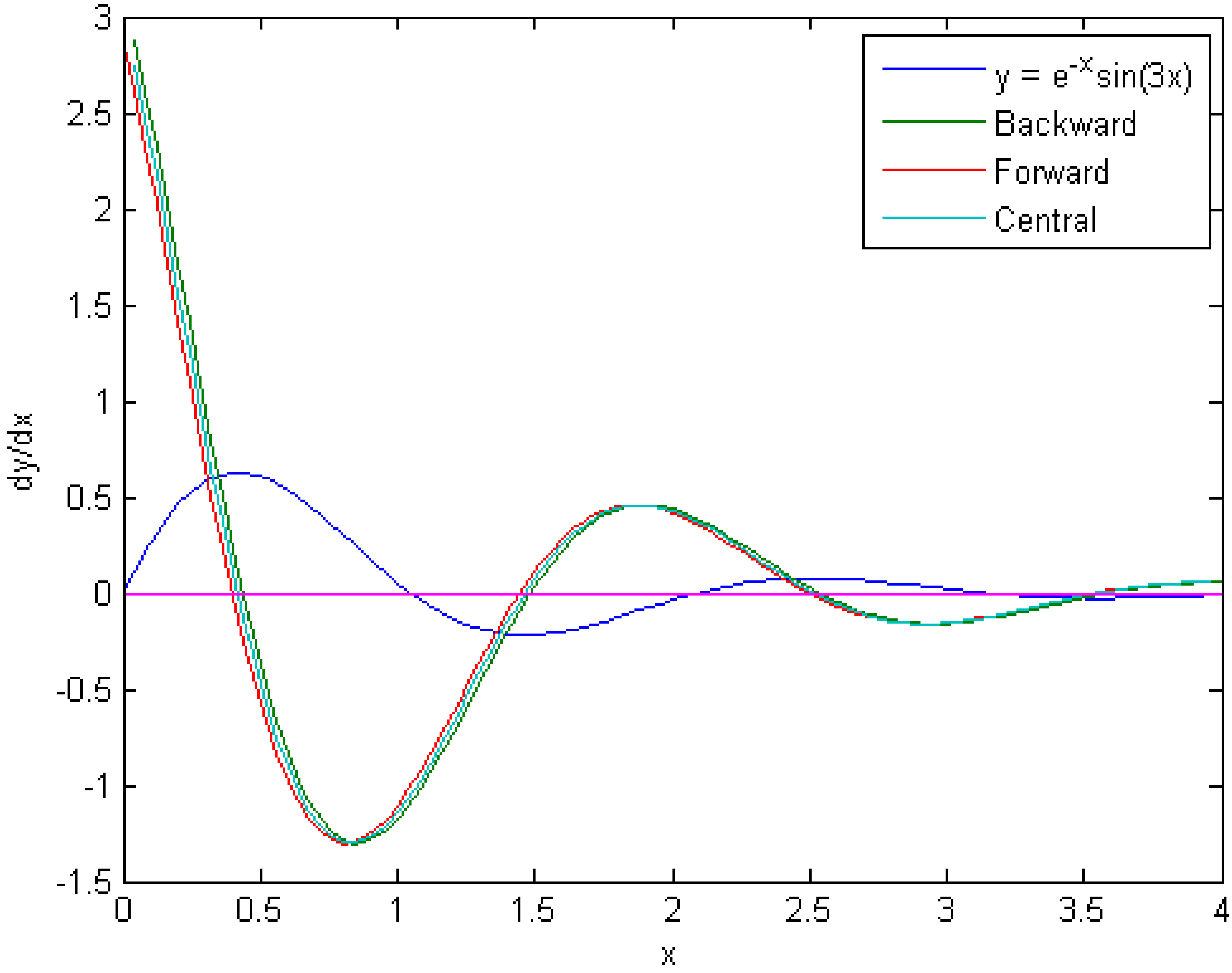


Problem 9.17: Scott Thomas
4
3
2
1
0
-1
-2
-3
(dy/dx)
0
1
2
3
4
5
6
7
8
9
10
x
Backward Difference
Forward Difference
Central Difference

Problem 9.19:

Compare the performance of the forward, backward, and central difference methods for estimating the derivative of $y(x) = e^{-x}\sin(3x)$. Use 101 points from $x = 0$ to $x = 4$.

Problem 9.19:



y = e^-x sin(3x)
Backward
Forward
Central
dy/dx
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5
4