Chapter 9b: Numerical Methods for Calculus and Differential Equations

- Initial-Value Problems
- Euler Method
- Time-Step Independence
- MATLAB ODE Solvers

Initial-Value Problems

Consider a skydiver falling from an airplane. A **Free-Body Diagram** of the skydiver is shown:

Newton's First Law is given by:

$$\sum F = ma$$

$$mg - F_D = m\frac{dv}{dt}$$

Substitute an expression for the **Aerodynamic Drag Force**:

$$mg - \frac{1}{2}\rho v^2 A C_D = m\frac{dv}{dt}$$

Initial-Value Problems

$$mg - \frac{1}{2}\rho v^2 A C_D = m\frac{dv}{dt}$$

This is a **First-Order Ordinary Differential Equation**. In particular, it is called an **Initial-Value Problem**, because it is solved by knowing an **Initial Value** of the **Dependent Variable**. For instance, we can assume that the **Downward Velocity** of the skydiver was initially zero:

$$v = 0 \text{ at } t = 0$$

Euler Method

$$mg - \frac{1}{2}\rho v^2 A C_D = m\frac{dv}{dt};\ v = 0 \text{ at } t = 0$$

This **Initial-Value Problem** can be solved for the skydiver's velocity as a function of time by using the **Euler Method**, which starts with the **Definition** of the **Derivative**. The derivative of the velocity is:

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

The derivative of the velocity can be approximated by allowing Δt be a small (but finite) value:

$$\frac{dv}{dt} \cong \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

Euler Method

$$\frac{dv}{dt} = \frac{1}{m}\left(mg - \frac{1}{2}\rho v^2 A C_D\right)$$

$$\frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{1}{m}\left(mg - \frac{1}{2}\rho v^2 A C_D\right)$$

$$v(t+\Delta t) = v(t) + \frac{\Delta t}{m}\left(mg - \frac{1}{2}\rho [v(t)]^2 A C_D\right)$$

Knowing the **Initial Condition** for the velocity, the skydiver's velocity can now be found by **Marching Forward in Time**.

Euler Method

$$v(t+\Delta t) = v(t) + \frac{\Delta t}{m}\left(mg - \frac{1}{2}\rho[v(t)]^2 AC_D\right)$$

This equation can be cast into a form appropriate for solution using MATLAB. The new velocity at time $t + \Delta t$ is:

$$v_{k+1} = v_k + \frac{\Delta t}{m}\left(mg - \frac{1}{2}\rho(v_k)^2 AC_D\right)$$

The new position at time $t + \Delta t$ is found by integrating the velocity:

$$x_{k+1} = x_k + \frac{\Delta t}{2}(v_k + v_{k+1})$$

The new time is found by incrementing by time step Δt:

$$t_{k+1} = t_k + \Delta t$$

Euler Method

Let $\Delta t = 0.1$ sec, $C_D = 0.8$, $A = 0.4\ \mathrm{m}^2$, $\rho = 1.225\ \mathrm{kg/m^3}$, $m = 82$ kg, $g = 9.81\ \mathrm{m/s^2}$

Initial Velocity: $v(t = 0) = 0$ or $v(k = 1) = 0$

Initial Position: $x(t = 0) = 0$ or $x(k = 1) = 0$

$$v_{k+1} = v_k + \frac{\Delta t}{m}\left(mg - \frac{1}{2}\rho(v_k)^2 A C_D\right)$$

$$v(2) = (0) + \frac{(0.1)}{(82)}\left((82)(9.81) - \frac{1}{2}(1.225)(0)^2(0.4)(0.8)\right)$$
$$= 0.981\ \mathrm{m/s}$$

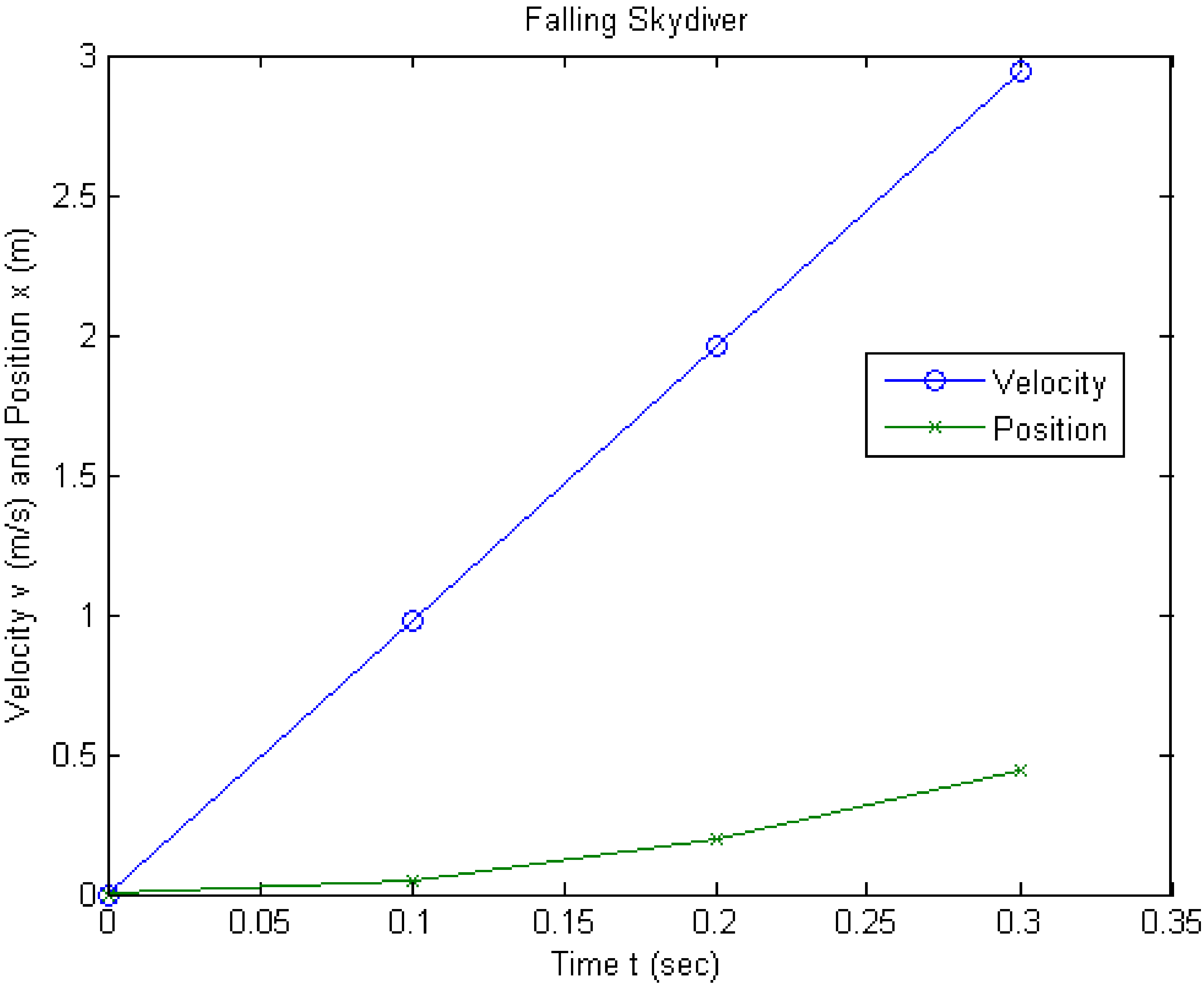
$$x(2) = (0) + \frac{(0.1)}{2}[(0) + (0.981)] = 0.04905\ \mathrm{m}$$

$$t(2) = 0 + 0.1 = 0.1\ \mathrm{sec}$$

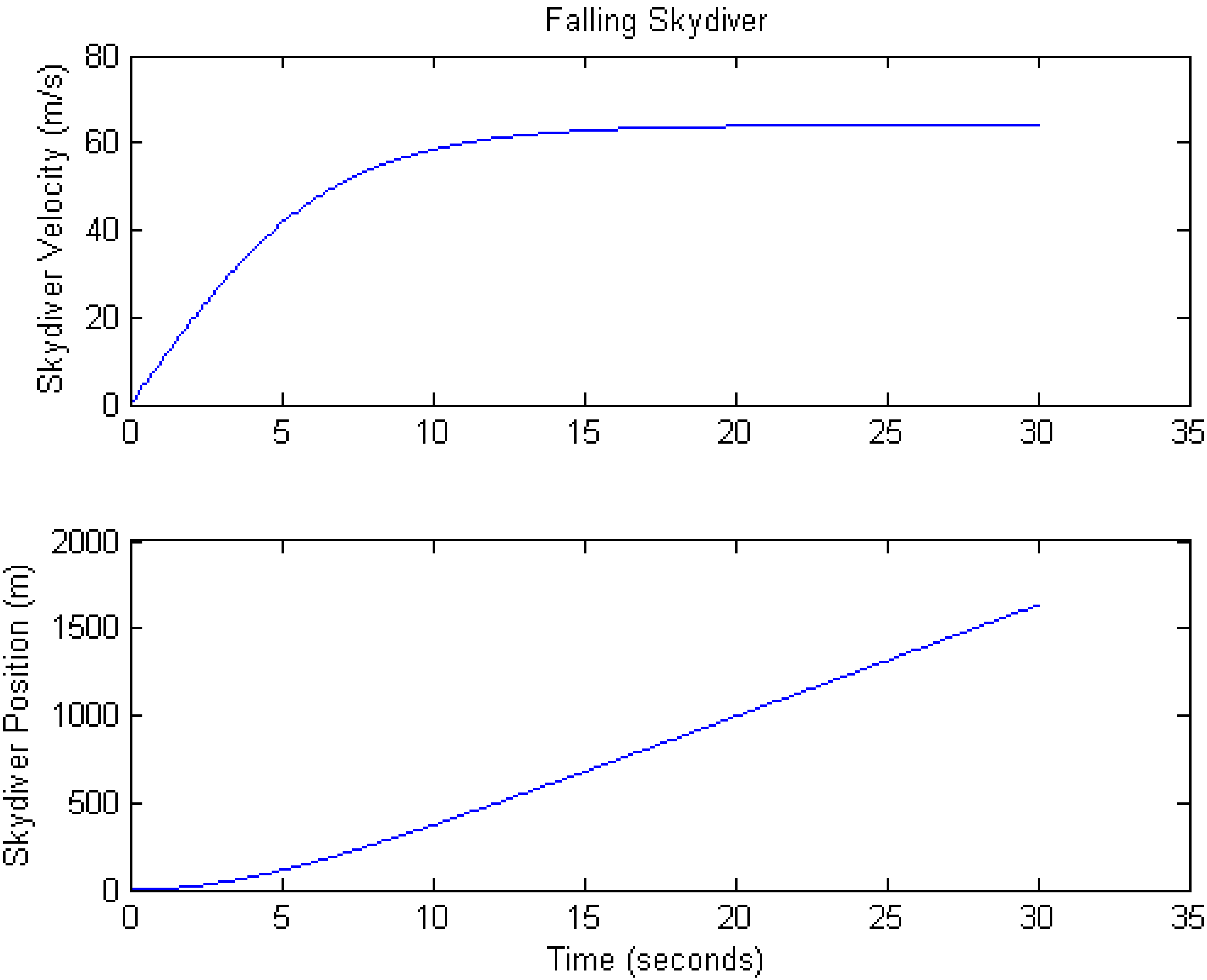
Euler Method

```
% Falling Skydiver: Euler Method
CD = 0.8; % Coefficient of Drag of the Skydiver's Body (Dimensionless)
A = 0.4;% Projected Area of the Skydiver's Body, m^2
rho = 1.225;% Density of Air, kg/m^3
m = 82;% Mass of Skydiver, kg
g = 9.81;% Acceleration due to Gravity, m/s^2
N = 3;
delta_t = 0.1;
t(1) = 0;% Initial Time, s
x(1) = 0;% Initial Position, m
v(1) = 0;% Initial Velocity, m/s
for k = 1:N
    v(k+1) = v(k) + delta_t/m*(m*g - 0.5*rho*v(k)^2*A*CD);
    x(k+1) = x(k) + delta_t/2*(v(k) + v(k+1));
    t(k+1) = t(k) + delta_t;
end
```

Euler Method

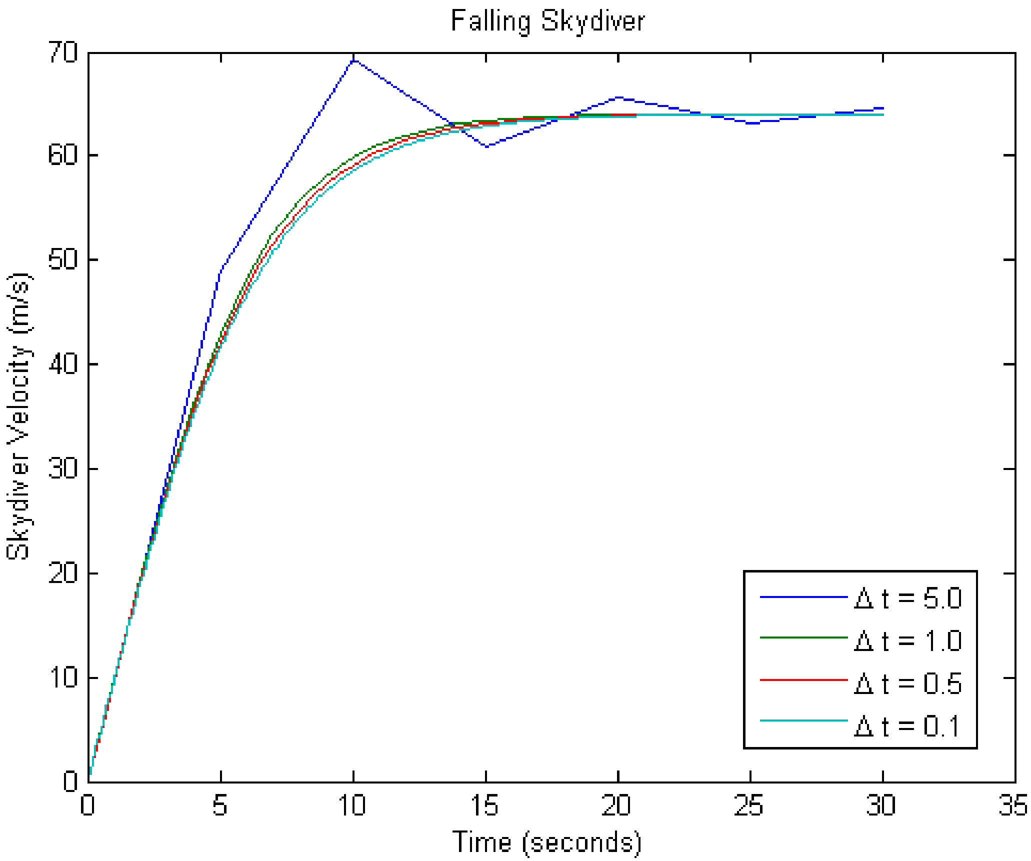


Euler Method



Time-Step Independence

The solution of the differential equation for the skydiver is dependent on the chosen time step Δt. As the time step size decreases, the solution curves begin to overlap. This is called **Time-Step Independence**. Conversely, if Δt becomes too large, the solution can become **unstable**, as shown for $\Delta t = 5.0$ seconds.



MATLAB ODE Solvers

- The **Euler Method** uses a fixed time step size that we specify and control.
- MATLAB has **built-in ODE solvers** that use variable step sizes. This speeds up the solution time. However, you no longer have control of the time step size.

`ode45`: Combination of 4th- and 5th-order Runge-Kutta methods.

`ode15s`: Used when `ode45` has difficulty.

- Basic syntax:

```
[t, y] = ode45(@ydot,tspan,y0)
```

MATLAB ODE Solvers

```
[t, y] = ode45(@ydot,tspan,y0)
```

`@ydot:` Handle of function file that describes ODE equation.

`tspan:` Starting and ending values of time *t* `[t0,tfinal]`

`y0:` Initial value of $y(0)$

Use MATLAB to compute and plot the solution of the following equation:

$$10\frac{dy}{dt} + y = 20 + 7\sin(2t) \qquad y(0) = 15$$

MATLAB ODE Solvers

$$10\frac{dy}{dt} + y = 20 + 7\sin(2t) \qquad y(0) = 15$$

$$\dot{y} = -\frac{1}{10}y + \frac{20}{10} + \frac{7}{10}\sin(2t) = -0.1y + 2 + 0.7\sin(2t)$$

Function File:

equation.m   T9_3_1.m

```
function [ ydot ] = equation( t,y )
% page 387, T9.3-1

ydot = -y/10 + 2 + 0.7*sin(2*t);

end
```

MATLAB ODE Solvers

$$\dot{y} = -0.1y + 2 + 0.7\sin(2t) \qquad y(0) = 15$$

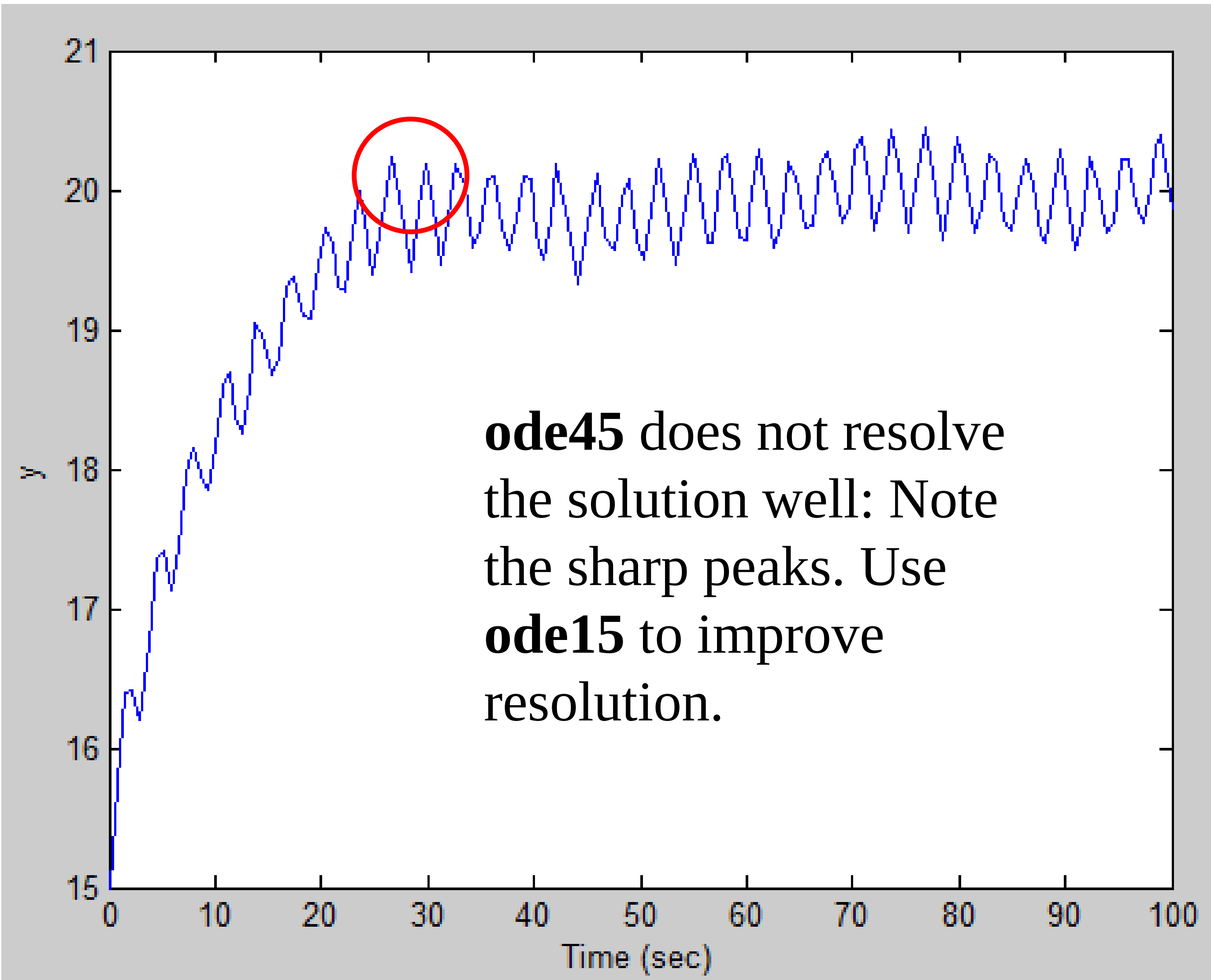
Script File: use ode45

equation.m × T9_3_1.m ×

```
% page 387, T9.3-1

clc
clear

[t,y] = ode45(@equation, [0 100], 15);

plot(t,y), xlabel('Time (sec)'), ylabel('y')
```

MATLAB ODE Solvers

$$\dot{y} = -0.1y + 2 + 0.7\sin(2t) \qquad y(0) = 15$$



ode45 does not resolve the solution well: Note the sharp peaks. Use **ode15** to improve resolution.

MATLAB ODE Solvers

$$\dot{y} = -0.1y + 2 + 0.7\sin(2t) \qquad y(0) = 15$$

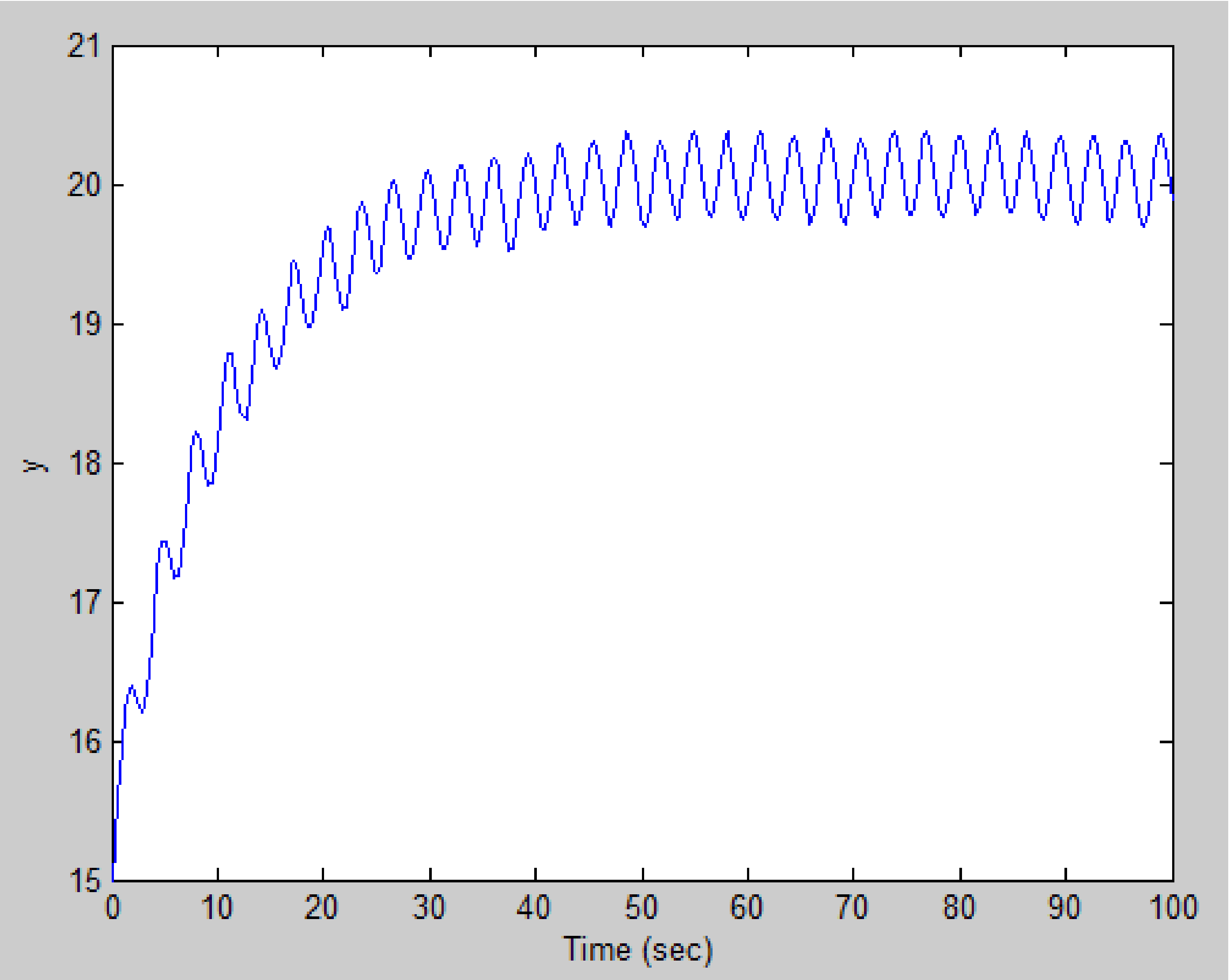
Script File: Use `ode15s`

equation.m　T9_3_1.m

```
% page 387, T9.3-1

clc
clear

[t,y] = ode15s(@equation, [0 100], 15);

plot(t,y), xlabel('Time (sec)'), ylabel('y')
```

MATLAB ODE Solvers

$$\dot{y} = -0.1y + 2 + 0.7\sin(2t) \qquad y(0) = 15$$



Problem 9.22:

Using the **Euler Method**, find the solution of the equation

$$6\dot{y} + y = f(t)$$

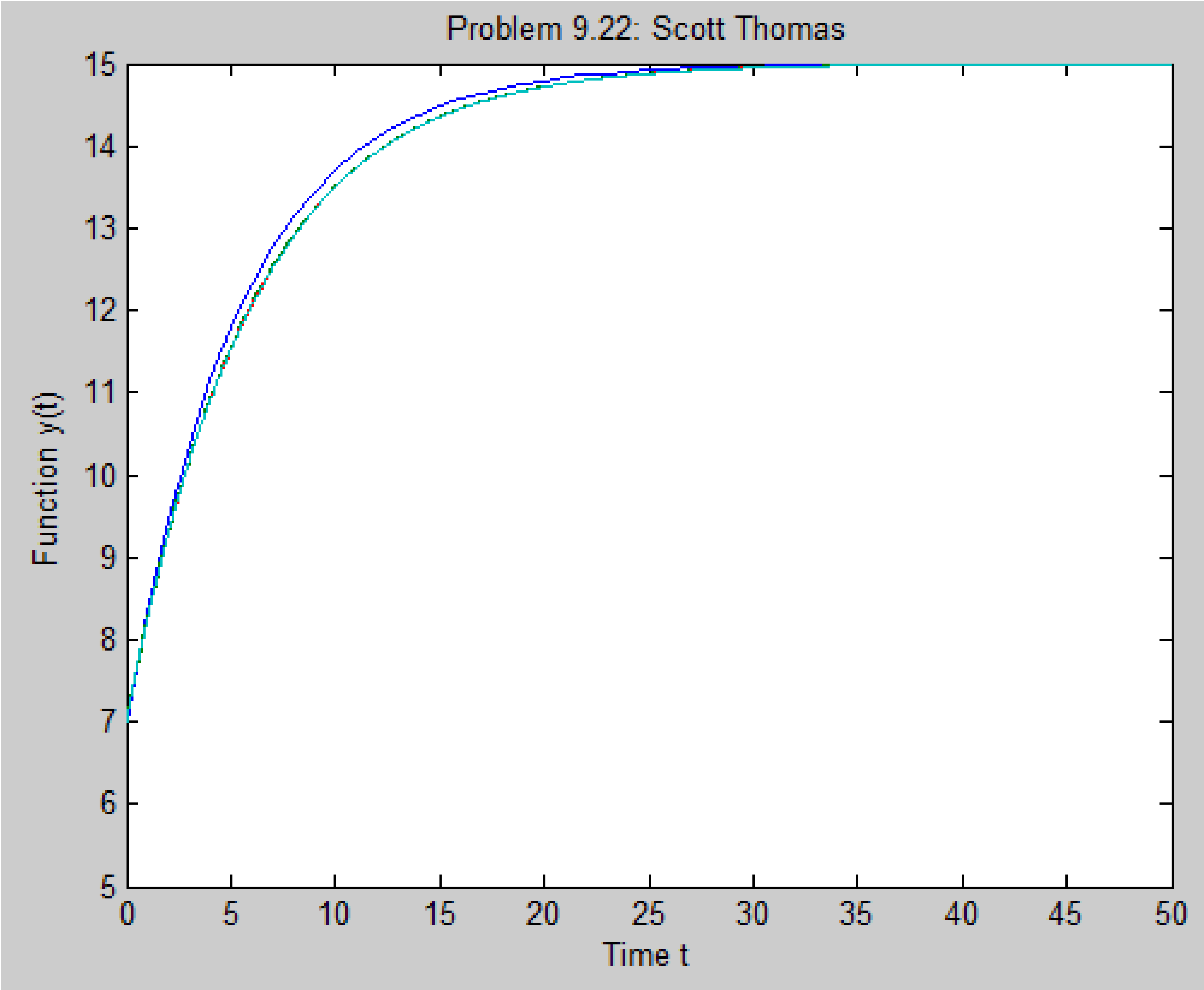
if $f(t) = 0$ for $t < 0$ and $f(t) = 15$ for $t \geq 0$. The initial condition is $y(0) = 7$.

The **Exact Solution** is obtained by using the **Integrating Factor Method**:

$$y(t) = 7e^{-t/6} + 15\left(1 - e^{-t/6}\right)$$

Plot the **Exact Solution** and the **Euler Method Solution** on the same graph to prove that your solution is **Time-Step Independent**.

Problem 9.22:



Problem 9.22: Scott Thomas
15
14
13
12
11
10
9
8
7
6
5
0
5
10
15
20
25
30
35
40
45
50
Time t
Function y(t)

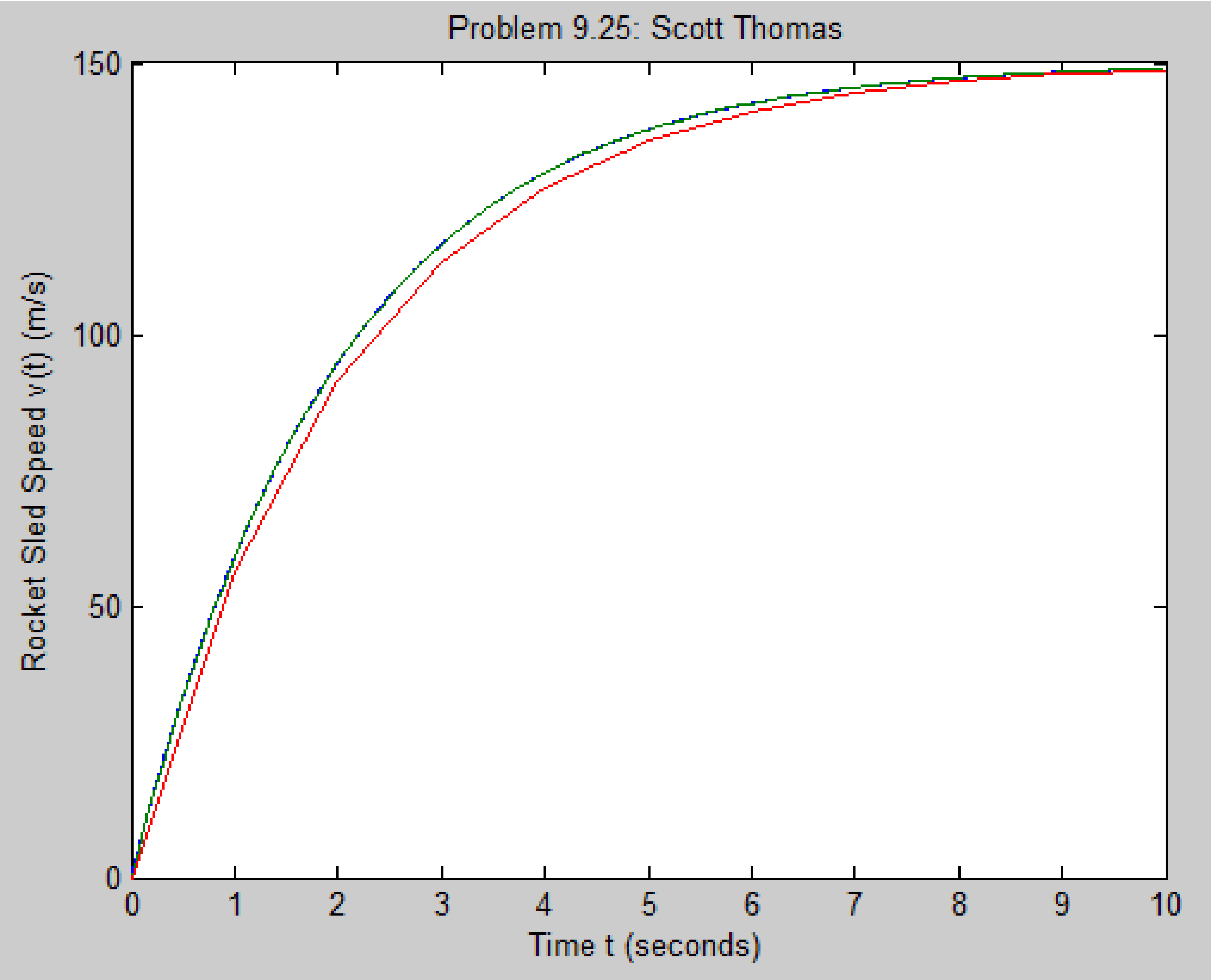
Problem 9.25:

The equation of motion of a rocket-propelled sled is, from Newton's Law,

$$m\dot{v} = f - cv$$

where $m = 1000$ kg is the sled mass, $f = 75{,}000$ N for $t > 0$ is the rocket thrust, and $c = 500$ N-s/m is an air resistance coefficient. Suppose that the initial velocity is $v(0) = 0$. Using the **Euler Method**, determine the speed of the sled until $t = 10$ seconds. Plot the sled velocity versus time, and show that the solution is independent of time step size.

Problem 9.25:


Problem 9.25: Scott Thomas
Rocket Sled Speed v(t) (m/s)
Time t (seconds)
0
50
100
150
0
1
2
3
4
5
6
7
8
9
10