Problem 9.28:

28. The equation describing the water height h in a spherical tank with a drain at the bottom is

$$\pi(2rh - h^2)\frac{dh}{dt} = -C_d A\sqrt{2gh}$$

Suppose the tank's radius is $r = 3$ m and the circular drain hole has a radius of 2 cm. Assume that $C_d = 0.5$ and that the initial water height is $h(0) = 5$ m. Use $g = 9.81$ m/s^2.

a. Use an approximation to estimate how long it takes for the tank to empty.

b. Plot the water height as a function of time until $h(t) = 0$.

Use the Predictor-Corrector Method to solve this problem.

Problem setup:

Use the Predictor-Corrector method to solve this problem. Solve the differential equation for dh/dt:

$$\frac{dh}{dt} = \frac{-C_d A\sqrt{2gh}}{\pi(2rh - h^2)} = g(t,h)$$

Approximate the derivative as follows:

$$\frac{dh}{dt} \approx \frac{\big(h(t+\Delta t) - h(t)\big)}{\Delta t} = g(t,h)$$

where Δt is the time step size. Solve for $h(t+\Delta t)$:

$$h(t+\Delta t) = h(t) + g(t,h)\Delta t$$

Put this equation into a form appropriate for computer solution. This is called the Difference Equation:

$$h(t_{k+1}) = h(t_k) + g[t_k, h(t_k)]\Delta t; \quad \text{where} \quad t_{k+1} = t_k + \Delta t$$

Rewrite this equation as follows. This is the Euler Predictor Difference Equation:

$$x_{k+1} = h_k + g(t_k, h_k)\Delta t$$

This represents a preliminary estimate for h_{k+1}, which is designated as x_{k+1} to avoid confusion. This estimate is then corrected using the Trapezoidal Corrector Difference Equation:

$$h_{k+1} = h_k + \frac{\Delta t}{2}[g(t_k, h_k) + g(t_{k+1}, x_{k+1})]$$

Substitute the expression for $g(t,h)$ into the difference equations:

$$g(t,h) = \frac{-C_d A\sqrt{2gh}}{\pi(2rh - h^2)}$$

$$g(t_k, h_k) = \frac{-C_d A\sqrt{2gh_k}}{\pi(2rh_k - h_k^2)}$$

$$g(t_{k+1}, x_{k+1}) = \frac{-C_d A\sqrt{2gx_{k+1}}}{\pi(2rx_{k+1} - x_{k+1}^2)}$$

Predictor:

$$x_{k+1} = h_k + g(t_k, h_k)\Delta t$$

Corrector:

$$h_{k+1} = h_k + \frac{\Delta t}{2}[g(t_k, h_k) + g(t_{k+1}, x_{k+1})]$$

In the problem statement, the initial condition is $h(0) = 5.0 \text{ m}$, $r_{\text{TANK}} = 3.0 \text{ m}$, $r_{\text{DRAIN}} = 2.0 \text{ cm} = 0.02 \text{ m}$. Let $\Delta t = 100.0$. The order in which calculations are made is important, as shown below.

<u>For $k = 1$:</u>

$$h_1 = 5.0$$

$$t_1 = 0.0$$

$$A = \pi r_{DRAIN}^2 = \pi(0.02)^2 = 1.2566 \times 10^{-3}$$

$$g(t_1, h_1) = \frac{-C_d A\sqrt{2gh_1}}{\pi(2rh_1 - h_1^2)} = \frac{-(0.5)(1.2566 \times 10^{-3})\sqrt{2(9.81)(5.0)}}{\pi[2(3.0)(5.0) - (5.0)^2]} = -3.9617 \times 10^{-4}$$

Predictor:

$$x_2 = h_1 + g(t_1, h_1)\Delta t = (5.0) + (-3.9617 \times 10^{-4})(100.0) = 4.9604$$

Corrector:

$$t_2 = t_1 + \Delta t = 0.0 + 100.0 = 100.0$$

$$g(t_2, x_2) = \frac{-C_d A\sqrt{2gx_2}}{\pi(2rx_2 - x_2^2)} = \frac{-(0.5)(1.2566 \times 10^{-3})\sqrt{2(9.81)(4.9604)}}{\pi[2(3.0)(4.9604) - (4.9604)^2]} = -3.8260 \times 10^{-4}$$

$$h_2 = h_1 + \frac{\Delta t}{2}[g(t_1, h_1) + g(t_2, x_2)] = (5.0) + \frac{(100.0)}{2}[(-3.9617 \times 10^{-4}) + (-3.8260 \times 10^{-4})] = 4.9611$$

<u>For $k = 2$:</u>

$$g(t_2, h_2) = \frac{-C_d A\sqrt{2gh_2}}{\pi(2rh_2 - h_2^2)} = \frac{-(0.5)(1.2566 \times 10^{-3})\sqrt{2(9.81)(4.9611)}}{\pi[2(3.0)(4.9611) - (4.9611)^2]} = -3.8283 \times 10^{-4}$$

Predictor:

$$x_3 = h_2 + g(t_2, h_2)\Delta t = (4.9611) + (-3.8283 \times 10^{-4})(100.0) = 4.9228$$

Corrector:

$$t_3 = t_2 + \Delta t = 100.0 + 100.0 = 200.0$$

$$g(t_3, x_3) = \frac{-C_d A\sqrt{2gx_3}}{\pi(2rx_3 - x_3^2)} = \frac{-(0.5)(1.2566 \times 10^{-3})\sqrt{2(9.81)(4.9228)}}{\pi[2(3.0)(4.9228) - (4.9228)^2]} = -3.7065 \times 10^{-4}$$

$$h_3 = h_2 + \frac{\Delta t}{2}[g(t_2, h_2) + g(t_3, x_3)] = (4.9611) + \frac{100.0}{2}[(-3.8283 \times 10^{-4}) + (-3.7065 \times 10^{-4})]$$
$$= 4.9234$$

<u>For $k = 3$:</u>

$$g(t_3, h_3) = \frac{-C_d A\sqrt{2gh_3}}{\pi(2rh_3 - h_3^2)} = \frac{-(0.5)(1.2566 \times 10^{-3})\sqrt{2(9.81)(4.9234)}}{\pi[2(3.0)(4.9234) - (4.9234)^2]} = -3.7083 \times 10^{-4}$$

Predictor:

$$x_4 = h_3 + g(t_3, h_3)\Delta t = (4.9234) + (-3.7083 \times 10^{-4})(100.0) = 4.8863$$

Corrector:

$$t_4 = t_3 + \Delta t = 200.0 + 100.0 = 300.0$$

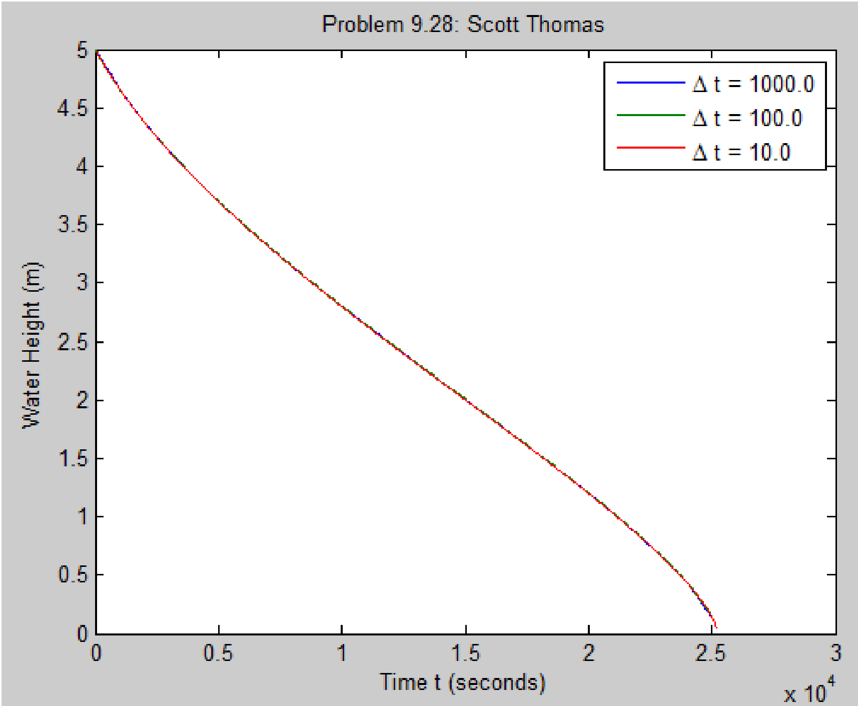
$$g(t_4, x_4) = \frac{-C_d A\sqrt{2gx_4}}{\pi(2rx_4 - x_4^2)} = \frac{-(0.5)(1.2566 \times 10^{-3})\sqrt{2(9.81)(4.8863)}}{\pi[2(3.0)(4.8863) - (4.8863)^2]} = -3.5984 \times 10^{-4}$$

$$h_4 = h_3 + \frac{\Delta t}{2}[g(t_3, h_3) + g(t_4, x_4)] = (4.9234) + \frac{100.0}{2}[(-3.7083 \times 10^{-4}) + (-3.5984 \times 10^{-4})]$$
$$= 4.8869$$

```
% Problem 9.28
clear
clc
disp('Problem 9.28: Scott Thomas')

r = 3.0;% m
rdrain = 0.02; % m
g = 9.81; %m/s^2
Cd = 0.5;
A = pi*rdrain^2;

N1 = 10;
delta_t1 = 100.0;
h1 = zeros(1,N1);
t1 = zeros(1,N1);
h1(1) = 5.0;
for k = 1:N1
    g1(k) = -Cd*A*sqrt(2*g*h1(k))/(pi*(2*r*h1(k) - h1(k)^2));
%    disp('Predictor')
    x1(k+1) = h1(k) + g1(k)*delta_t1;
    t1(k+1) = t1(k) + delta_t1;
    g1(k+1) = -Cd*A*sqrt(2*g*x1(k+1))/(pi*(2*r*x1(k+1) - x1(k+1)^2));
%    disp('Corrector')
    h1(k+1) = h1(k) + delta_t1/2*(g1(k) + g1(k+1));
%    disp('End of Process')
end
x1
h1
```

```
Problem 9.28: Scott Thomas

x1 =

         0    4.9604    4.9228    4.8863    4.8509    4.8163    4.7827    4.7498    4.7177    4.6862    4.6554


h1 =

    5.0000    4.9611    4.9234    4.8869    4.8514    4.8168    4.7831    4.7502    4.7180    4.6865    4.6557

>> 
```

```
% Problem 9.28
clear
clc
disp('Problem 9.28: Scott Thomas')

r = 3.0;% m
rdrain = 0.02; % m
g = 9.81; %m/s^2
Cd = 0.5;
A = pi*rdrain^2;

N1 = 252;
delta_t1 = 100.0;
h1 = zeros(1,N1);
t1 = zeros(1,N1);
h1(1) = 5.0;
for k = 1:N1
    g1(k) = -Cd*A*sqrt(2*g*h1(k))/(pi*(2*r*h1(k) - h1(k)^2));
%     disp('Predictor')
    x1(k+1) = h1(k) + g1(k)*delta_t1;
    t1(k+1) = t1(k) + delta_t1;
    g1(k+1) = -Cd*A*sqrt(2*g*x1(k+1))/(pi*(2*r*x1(k+1) - x1(k+1)^2));
%     disp('Corrector')
    h1(k+1) = h1(k) + delta_t1/2*(g1(k) + g1(k+1));
%     disp('End of Process')
end

N2 = 25;
delta_t2 = 1000.0;
h2 = zeros(1,N2);
t2 = zeros(1,N2);
h2(1) = 5.0;
for k = 1:N2
    g2(k) = -Cd*A*sqrt(2*g*h2(k))/(pi*(2*r*h2(k) - h2(k)^2));
%     disp('Predictor')
    x2(k+1) = h2(k) + g2(k)*delta_t2;
    t2(k+1) = t2(k) + delta_t2;
    g2(k+1) = -Cd*A*sqrt(2*g*x2(k+1))/(pi*(2*r*x2(k+1) - x2(k+1)^2));
%     disp('Corrector')
    h2(k+1) = h2(k) + delta_t2/2*(g2(k) + g2(k+1));
%     disp('End of Process')
end

N3 = 2520;
delta_t3 = 10.0;
h3 = zeros(1,N3);
t3 = zeros(1,N3);
h3(1) = 5.0;
for k = 1:N3
    g3(k) = -Cd*A*sqrt(2*g*h3(k))/(pi*(2*r*h3(k) - h3(k)^2));
%     disp('Predictor')
    x3(k+1) = h3(k) + g3(k)*delta_t3;
    t3(k+1) = t3(k) + delta_t3;
    g3(k+1) = -Cd*A*sqrt(2*g*x3(k+1))/(pi*(2*r*x3(k+1) - x3(k+1)^2));
%     disp('Corrector')
    h3(k+1) = h3(k) + delta_t3/2*(g3(k) + g3(k+1));
%     disp('End of Process')
end

plot(t2,h2, t1,h1, t3,h3 )
xlabel('Time t (seconds)'), ylabel('Water Height (m)')
title('Problem 9.28: Scott Thomas')
legend('\Delta t = 1000.0','\Delta t = 100.0','\Delta t = 10.0','Location', 'NorthEast')
%axis([0 5 0 30])

```



Problem 9.28: Scott Thomas
Δ t = 1000.0
Δ t = 100.0
Δ t = 10.0
Water Height (m)
Time t (seconds)
x 10^4
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5