ME 1020 Engineering Programming with MATLAB

Problem 9.29:

29. The following equation describes a certain dilution process, where $y(t)$ is the concentration of salt in a tank of freshwater to which salt brine is being added.

$$\frac{dy}{dt} + \frac{5}{10 + 2t}y = 4$$

Suppose that $y(0) = 0$. Plot $y(t)$ for $0 \le t \le 10$.

Problem setup:

Use the Predictor-Corrector method to solve this problem. Solve the differential equation for dy/dt:

$$\frac{dy}{dt} = 4 - \left(\frac{5y}{10 + 2t}\right) = g(t, y)$$

Approximate the derivative as follows:

$$\frac{dy}{dt} \approx \frac{\big(y(t + \Delta t) - y(t)\big)}{\Delta t} = g(t, y)$$

where Δt is the time step size. Solve for $y(t + \Delta t)$:

$$y(t + \Delta t) = y(t) + g(t, y)\Delta t$$

Put this equation into a form appropriate for computer solution. This is called the Difference Equation:

$$y(t_{k+1}) = y(t_k) + g[t_k, y(t_k)]\Delta t; \quad \text{where} \quad t_{k+1} = t_k + \Delta t$$

Rewrite this equation as follows. This is the Euler Predictor Difference Equation:

$$x_{k+1} = y_k + g(t_k, y_k)\Delta t$$

This represents a preliminary estimate for y_{k+1}, which is designated as x_{k+1} to avoid confusion. This estimate is then corrected using the Trapezoidal Corrector Difference Equation:

$$y_{k+1} = y_k + \frac{\Delta t}{2}[g(t_k, y_k) + g(t_{k+1}, x_{k+1})]$$

Substitute the expression for $g(t, y)$ into the difference equations:

$$g(t, y) = 4 - \left(\frac{5y}{10 + 2t}\right)$$

$$g(t_k, y_k) = 4 - \left(\frac{5y_k}{10 + 2t_k}\right)$$

$$g(t_{k+1}, x_{k+1}) = 4 - \left(\frac{5x_{k+1}}{10 + 2t_{k+1}}\right)$$

Predictor:

$$x_{k+1} = y_k + g(t_k, y_k)\Delta t$$

Corrector:

$$y_{k+1} = y_k + \frac{\Delta t}{2}[g(t_k, y_k) + g(t_{k+1}, x_{k+1})]$$

In the problem statement, the initial condition is $y(0) = 0.0$. Let $\Delta t = 0.01$. The order in which calculations are made is important, as shown below.

For $k = 1$:

$$y_1 = 0.0$$

$$t_1 = 0.0$$

$$g(t_1, y_1) = 4 - \left(\frac{5y_1}{10 + 2t_1}\right) = 4 - \left(\frac{5(0.0)}{10 + 2(0.0)}\right) = 4.0$$

Predictor:

$$x_2 = y_1 + g(t_1, y_1)\Delta t = (0.0) + (4.0)(0.01) = 0.04$$

Corrector:

$$t_2 = t_1 + \Delta t = 0.0 + 0.01 = 0.01$$

$$g(t_2, x_2) = 4 - \left(\frac{5x_2}{10 + 2t_2}\right) = 4 - \left(\frac{5(0.04)}{10 + 2(0.01)}\right) = 3.9800$$

$$y_2 = y_1 + \frac{\Delta t}{2}[g(t_1, y_1) + g(t_2, x_2)] = (0.0) + \frac{(0.01)}{2}[(4.0) + (3.9800)] = 0.039900$$

For $k = 2$:

$$g(t_2, y_2) = 4 - \left(\frac{5y_2}{10 + 2t_2}\right) = 4 - \left(\frac{5(0.039900)}{10 + 2(0.01)}\right) = 3.9801$$

Predictor:

$$x_3 = y_2 + g(t_2, y_2)\Delta t = (0.039900) + (3.9801)(0.01) = 0.079701$$

Corrector:

$$t_3 = t_2 + \Delta t = 0.01 + 0.01 = 0.02$$

$$g(t_3, x_3) = 4 - \left(\frac{5x_3}{10 + 2t_3}\right) = 4 - \left(\frac{5(0.079701)}{10 + 2(0.02)}\right) = 3.9603$$

$$y_3 = y_2 + \frac{\Delta t}{2}[g(t_2, y_2) + g(t_3, x_3)] = (0.039900) + \frac{0.01}{2}[(3.9801) + (3.9603)] = 0.079602$$

<u>For $k = 3$:</u>

$$g(t_3, y_3) = 4 - \left(\frac{5y_3}{10 + 2t_3}\right) = 4 - \left(\frac{5(0.079602)}{10 + 2(0.02)}\right) = 3.9603$$

Predictor:

$$x_4 = y_3 + g(t_3, y_3)\Delta t = (0.079602) + (3.9603)(0.01) = 0.11920$$

Corrector:

$$t_4 = t_3 + \Delta t = 0.02 + 0.01 = 0.03$$

$$g(t_4, x_4) = 4 - \left(\frac{5x_4}{10 + 2t_4}\right) = 4 - \left(\frac{5(0.11920)}{10 + 2(0.11920)}\right) = 3.9418$$

$$y_4 = y_3 + \frac{\Delta t}{2}[g(t_3, y_3) + g(t_4, x_4)] = (0.079602) + \frac{0.01}{2}[(3.9603) + (3.9418)] = 0.11911$$

```
% Problem 9.29
clear
clc
disp('Problem 9.29: Scott Thomas')

N1 = 10;
delta_t1 = 0.01;
y1 = zeros(1,N1);
t1 = zeros(1,N1);
y1(1) = 0.0;
for k = 1:N1
    g1(k) = 4 - 5*y1(k)/(10 + 2*t1(k));
%    disp('Predictor')
    x1(k+1) = y1(k) + g1(k)*delta_t1;
    t1(k+1) = t1(k) + delta_t1;
    g1(k+1) = 4 - 5*x1(k+1)/(10 + 2*t1(k+1));
%    disp('Corrector')
    y1(k+1) = y1(k) + delta_t1/2*(g1(k) + g1(k+1));
%    disp('End of Process')
end
t1
x1
y1
```

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t1 =

         0    0.0100    0.0200    0.0300    0.0400    0.0500    0.0600    0.0700    0.0800    0.0900    0.1000


x1 =

         0    0.0400    0.0797    0.1192    0.1585    0.1976    0.2366    0.2753    0.3138    0.3522    0.3904


y1 =

         0    0.0399    0.0796    0.1191    0.1584    0.1975    0.2365    0.2752    0.3137    0.3521    0.3903

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```
% Problem 9.29
clear
clc
disp('Problem 9.29: Scott Thomas')

N1 = 1000;
delta_t1 = 0.01;
y1 = zeros(1,N1);
t1 = zeros(1,N1);
y1(1) = 0.0;
for k = 1:N1
    g1(k) = 4 - 5*y1(k)/(10 + 2*t1(k));
%    disp('Predictor')
    x1(k+1) = y1(k) + g1(k)*delta_t1;
    t1(k+1) = t1(k) + delta_t1;
    g1(k+1) = 4 - 5*x1(k+1)/(10 + 2*t1(k+1));
%    disp('Corrector')
    y1(k+1) = y1(k) + delta_t1/2*(g1(k) + g1(k+1));
%    disp('End of Process')
end

N2 = 100;
delta_t2 = 0.1;
y2 = zeros(1,N2);
t2 = zeros(1,N2);
y2(1) = 0.0;
for k = 1:N2
    g2(k) = 4 - 5*y2(k)/(10 + 2*t2(k));
%    disp('Predictor')
    x2(k+1) = y2(k) + g2(k)*delta_t2;
    t2(k+1) = t2(k) + delta_t2;
    g2(k+1) = 4 - 5*x2(k+1)/(10 + 2*t2(k+1));
%    disp('Corrector')
    y2(k+1) = y2(k) + delta_t2/2*(g2(k) + g2(k+1));
%    disp('End of Process')
end

N3 = 10;
delta_t3 = 1.0;
y3 = zeros(1,N3);
t3 = zeros(1,N3);
```

```
y3(1) = 0.0;
for k = 1:N3
    g3(k) = 4 - 5*y3(k)/(10 + 2*t3(k));
%    disp('Predictor')
    x3(k+1) = y3(k) + g3(k)*delta_t3;
    t3(k+1) = t3(k) + delta_t3;
    g3(k+1) = 4 - 5*x3(k+1)/(10 + 2*t3(k+1));
%    disp('Corrector')
    y3(k+1) = y3(k) + delta_t3/2*(g3(k) + g3(k+1));
%    disp('End of Process')
end

plot(t3,y3, t2,y2, t1,y1)
xlabel('Time t (seconds)'), ylabel('Concentration of Salt')
title('Problem 9.29: Scott Thomas')
legend('\Delta t = 1.0','\Delta t = 0.1','\Delta t = 0.01','Location', 'SouthEast')
%axis([0 5 0 30])
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Problem 9.29: Scott Thomas