Problem 9.33:

33. Van der Pol's equation has been used to describe many oscillatory processes. It is

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0$$

Plot $y(t)$ for $\mu = 1$ and $0 \le t \le 20$, using the initial conditions $y(0) = 5$, $\dot{y}(0) = 0$.

This is a second-order ordinary differential equation. Rewrite the equation by solving for the second derivative.

$$\ddot{y} = \mu(1 - y^2)\dot{y} - y$$

$$\text{Let } \; x_1 = y \; \text{ and } \; x_2 = \dot{y}$$

Taking the derivative of the first equation gives

$$\dot{x}_1 = \dot{y} = x_2 \quad \text{or} \quad \dot{x}_1 = x_2$$

Taking the derivative of the second equation gives

$$\dot{x}_2 = \ddot{y} = \mu(1 - y^2)\dot{y} - y = \mu(1 - x_1^2)x_2 - x_1 \; \text{ or } \; \dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1$$

The initial conditions are

$$y(0) = x_1(0) = 5 \; \text{ and } \; \dot{y}(0) = x_2 = 0$$

The original second order ordinary differential equation is now converted into two first order ordinary differential equations that are coupled.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1, \quad x_1(0) = 5, \; x_2(0) = 0$$

Use the Euler method for this problem. The system of equations can be discretized as follows:

$$x_{1,k+1} = x_{1,k} + \Delta t \cdot x_{2,k}$$

$$x_{2,k+1} = x_{2,k} + \Delta t\left[\mu\left(1 - x_{1,k}^2\right)x_{2,k} - x_{1,k}\right] = x_{2,k} + \Delta t \cdot g(x_{1,k}, x_{2,k})$$

$$g\left(x_{1,k}, x_{2,k}\right) = \mu\left(1 - x_{1,k}^2\right)x_{2,k} - x_{1,k}$$

Let $\Delta t = 0.001$ and $\mu = 1.0$.

For $k = 1$:

$$x_{1,1} = 5.0, \quad x_{2,1} = 0.0$$

$$x_{1,2} = x_{1,1} + \Delta t \cdot x_{2,1} = (5.0) + (0.001)(0.0) = 5.0$$

$$g\left(x_{1,1}, x_{2,1}\right) = \mu\left(1 - x_{1,1}^2\right)x_{2,1} - x_{1,1} = (1.0)[1 - (5.0)^2](0.0) - (5.0) = -5.0$$

$$x_{2,2} = x_{2,1} + \Delta t\left[g\left(x_{1,1}, x_{2,1}\right)\right] = (0.0) + (0.001)[(-5.0)] = -0.005$$

$$t_2 = t_1 + \Delta t = 0.0 + 0.001 = 0.001$$

For $k = 2$:

$$x_{1,3} = x_{1,2} + \Delta t \cdot x_{2,2} = (5.0) + (0.001)(-0.005) = 4.999995$$

$$g\left(x_{1,2}, x_{2,2}\right) = \mu\left(1 - x_{1,2}^2\right)x_{2,2} - x_{1,2} = (1.0)[1 - (5.0)^2](-0.005) - (5.0) = -4.88$$

$$x_{2,3} = x_{2,2} + \Delta t\left[g\left(x_{1,2}, x_{2,2}\right)\right] = (-0.005) + (0.001)[(-4.88)] = -0.00988$$

$$t_3 = t_2 + \Delta t = 0.001 + 0.001 = 0.002$$

For $k = 3$:

$$x_{1,4} = x_{1,3} + \Delta t \cdot x_{2,3} = (4.999995) + (0.001)(-0.00988) = 4.999985$$

$$g\left(x_{1,3}, x_{2,3}\right) = \mu\left(1 - x_{1,3}^2\right)x_{2,3} - x_{1,3} = (1.0)[1 - (4.999995)^2](-0.00988) - (4.999995) = -4.762875$$

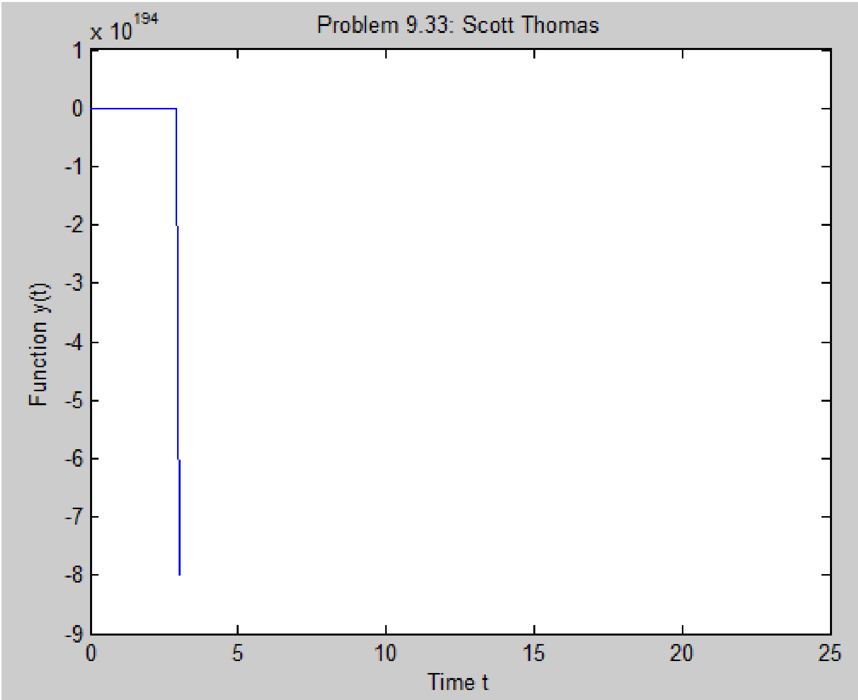
$$x_{2,4} = x_{2,3} + \Delta t\left[g\left(x_{1,1}, x_{2,1}\right)\right] = (-0.00988) + (0.001)[(-4.762875)] = -0.0146429$$

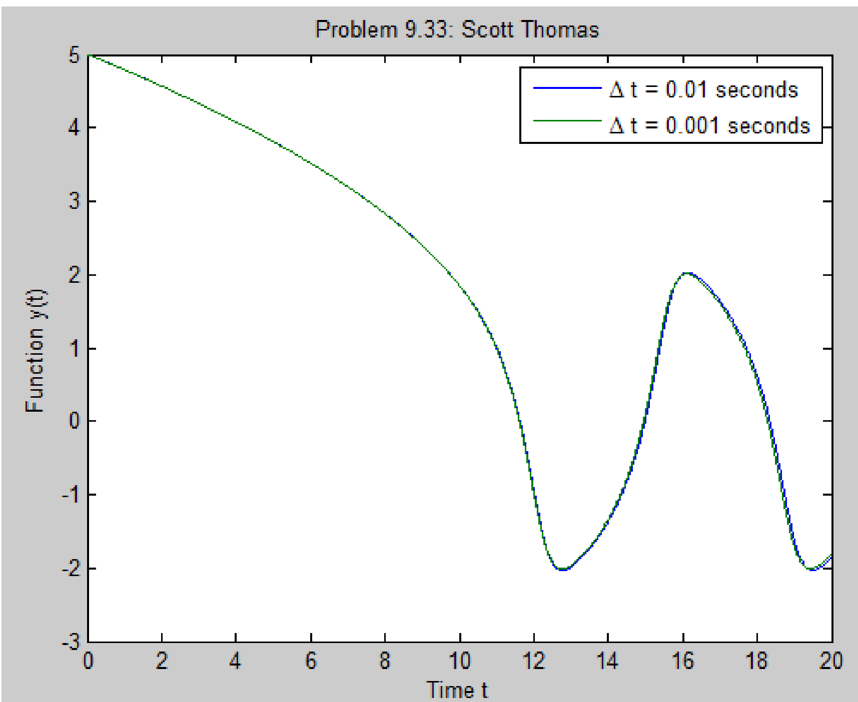
$$t_4 = t_3 + \Delta t = 0.002 + 0.001 = 0.003$$

```
% Problem 9.33
clear
clc
disp('Problem 9.33: Scott Thomas')

format long
mu = 1.0;
Na = 3;
delta_ta = 0.001;
x1a = zeros(1,Na);
x2a = zeros(1,Na);
g = zeros(1,Na);

ta = zeros(1,Na);
x1a(1) = 5.0;
x2a(1) = 0.0;

for k = 1:Na
    x1a(k+1) = x1a(k) + x2a(k)*delta_ta;
    g(k) = mu*(1 - x1a(k)^2)*x2a(k) - x1a(k);
    x2a(k+1) = x2a(k) + g(k)*delta_ta;
    ta(k+1) = ta(k) + delta_ta;
end
ta
x1a
x2a

plot(ta, x1a)
xlabel('Time t'), ylabel('Function y(t)')
title('Problem 9.33: Scott Thomas')
```

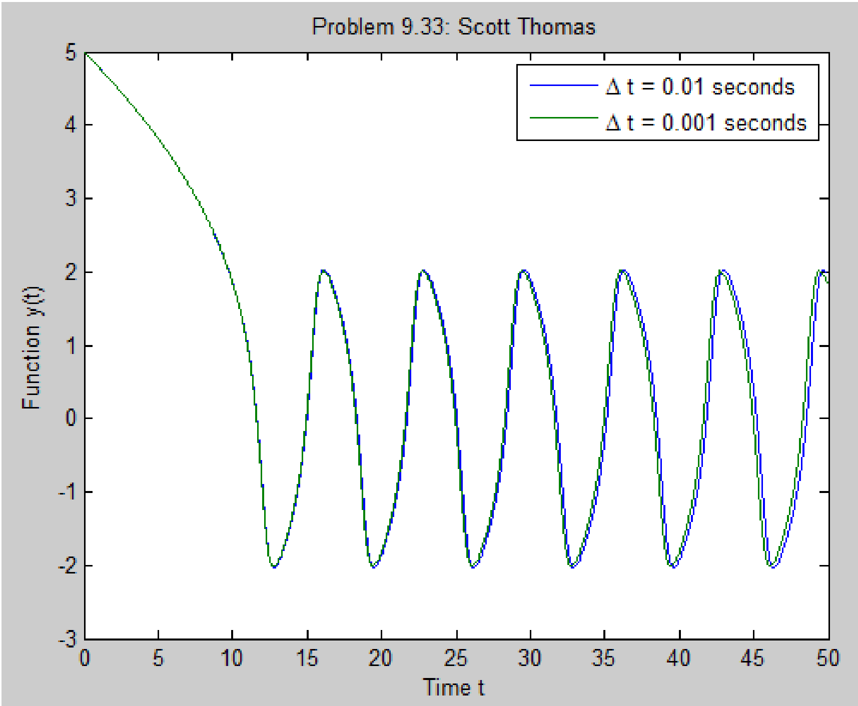
```
Problem 9.33: Scott Thomas

ta =

                   0   0.001000000000000   0.002000000000000   0.003000000000000


x1a =

   5.000000000000000   5.000000000000000   4.999995000000000   4.999985120000000


x2a =

                   0  -0.005000000000000  -0.009880000000000  -0.014642875494000

>>
```

Time step independence check: For $\Delta t = 0.1$, the solution is unstable.





The time step should be decreased as the final time increases:


Problem 9.33: Scott Thomas
Δ t = 0.01 seconds
Δ t = 0.001 seconds
Function y(t)
Time t
5
4
3
2
1
0
-1
-2
-3
0
5
10
15
20
25
30
35
40
45
50

```matlab
% Problem 9.33
clear
clc
disp('Problem 9.33: Scott Thomas')

format long
mu = 1.0;
Na = 2000;
delta_ta = 0.01;
x1a = zeros(1,Na);
x2a = zeros(1,Na);
ga = zeros(1,Na);

ta = zeros(1,Na);
x1a(1) = 5.0;
x2a(1) = 0.0;

for k = 1:Na
    x1a(k+1) = x1a(k) + x2a(k)*delta_ta;
    ga(k) = mu*(1 - x1a(k)^2)*x2a(k) - x1a(k);
    x2a(k+1) = x2a(k) + ga(k)*delta_ta;
    ta(k+1) = ta(k) + delta_ta;
end

Nb = 20000;
delta_tb = 0.001;
x1b = zeros(1,Nb);
x2b = zeros(1,Nb);
gb = zeros(1,Nb);

tb = zeros(1,Nb);
x1b(1) = 5.0;
x2b(1) = 0.0;

for k = 1:Nb
    x1b(k+1) = x1b(k) + x2b(k)*delta_tb;
    gb(k) = mu*(1 - x1b(k)^2)*x2b(k) - x1b(k);
    x2b(k+1) = x2b(k) + gb(k)*delta_tb;
    tb(k+1) = tb(k) + delta_tb;
end

plot(ta, x1a, tb, x1b)
xlabel('Time t'), ylabel('Function y(t)')
title('Problem 9.33: Scott Thomas')
legend('\Delta t = 0.01 seconds','\Delta t = 0.001 seconds','Location','NorthEast')
axis([0 20 -3 5])
```