ME 1020 Engineering Programming with MATLAB

Problem 9.34:

34. The equation of motion for a pendulum whose base is accelerating horizontally with an acceleration $a(t)$ is

$$L\ddot{\theta} + g\sin\theta = a(t)\cos\theta$$

Suppose that $g = 9.81$ m/s^2, $L = 1$ m, and $\dot{\theta}(0) = 0$. Plot $\theta(t)$ for $0 \leq t \leq 10$ s for the following three cases.

a. The acceleration is constant: $a = 5$ m/s^2, and $\theta(0) = 0.5$ rad.
b. The acceleration is constant: $a = 5$ m/s^2, and $\theta(0) = 3$ rad.
c. The acceleration is linear with time: $a = 0.5t$ m/s^2, and $\theta(0) = 3$ rad.

This is a second-order ordinary differential equation. Rewrite the equation by solving for the second derivative.

$$\ddot{\theta} = \frac{1}{L}[a(t)\cos\theta - g\sin\theta]$$

$$\text{Let}\ \ x_1 = \theta\ \text{ and }\ x_2 = \dot{\theta}$$

Taking the derivative of the first equation gives

$$\dot{x}_1 = \dot{\theta} = x_2 \quad \text{or} \quad \dot{x}_1 = x_2$$

Taking the derivative of the second equation gives

$$\dot{x}_2 = \ddot{y} = \frac{1}{L}[a(t)\cos\theta - g\sin\theta] = \frac{1}{L}[a(t)\cos x_1 - g\sin x_1] \ \text{ or }\ \dot{x}_2 = \frac{1}{L}[a(t)\cos x_1 - g\sin x_1]$$

The initial conditions are

$$\theta(0) = x_1(0) = 0.5 \ \text{ and }\ \dot{\theta}(0) = x_2 = 0$$

The original second order ordinary differential equation is now converted into two first order ordinary differential equations that are coupled.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{1}{L}[a(t)\cos x_1 - g\sin x_1], \quad x_1(0) = 0.5, \ \ x_2(0) = 0$$

Use the Euler method for this problem. The system of equations can be discretized as follows:

$$x_{1,k+1} = x_{1,k} + \Delta t \cdot x_{2,k}$$

$$x_{2,k+1} = x_{2,k} + \Delta t\left[\frac{1}{L}[a(t)\cos x_1 - g\sin x_1]\right] = x_{2,k} + \Delta t \cdot g(x_{1,k}, t_k)$$

$$g(x_{1,k}, t_k) = \frac{1}{L}\left[a(t_k)\cos x_{1,k} - g\sin x_{1,k}\right]$$

Let $\Delta t = 0.001$.

For $k = 1$:

$$x_{1,1} = 0.5, \quad x_{2,1} = 0.0$$

$$x_{1,2} = x_{1,1} + \Delta t \cdot x_{2,1} = (0.5) + (0.001)(0.0) = 0.5$$

$$g(x_{1,1}, t_1) = \frac{1}{L}\left[a(t_1)\cos x_{1,1} - g\sin x_{1,1}\right] = \frac{1}{(1.0)}[(5.0)\cos(0.5) - (9.81)\sin(0.5)] = -0.3152517$$

$$x_{2,2} = x_{2,1} + \Delta t\left[g(x_{1,1}, t_1)\right] = (0.0) + (0.001)(-0.3152517) = -3.152517 \times 10^{-4}$$

$$t_2 = t_1 + \Delta t = 0.0 + 0.001 = 0.001$$

For $k = 2$:

$$x_{1,3} = x_{1,2} + \Delta t \cdot x_{2,2} = (0.5) + (0.001)(-3.152517 \times 10^{-4}) = 0.4999996847$$

$$g(x_{1,2}, t_2) = \frac{1}{L}\left[a(t_2)\cos x_{1,2} - g\sin x_{1,2}\right] = \frac{1}{(1.0)}[(5.0)\cos(0.5) - (9.81)\sin(0.5)] = -0.3152517$$

$$x_{2,3} = x_{2,2} + \Delta t\left[g(x_{1,2}, t_2)\right] = (-3.152517 \times 10^{-4}) + (0.001)[(-0.3152517)] = -6.305034 \times 10^{-4}$$

$$t_3 = t_2 + \Delta t = 0.001 + 0.001 = 0.002$$

For $k = 3$:

$$x_{1,4} = x_{1,3} + \Delta t \cdot x_{2,3} = (0.4999996847) + (0.001)(-6.305034 \times 10^{-4}) = 0.4999990535$$

$$g(x_{1,3}, t_3) = \frac{1}{L}\left[a(t_3)\cos x_{1,3} - g\sin x_{1,3}\right] = \frac{1}{(1.0)}[(5.0)\cos(0.4999996847) - (9.81)\sin(0.4999996847)]$$
$$= -0.315248246$$

$$x_{2,4} = x_{2,3} + \Delta t\left[g(x_{1,3}, t_3)\right] = (-6.305034 \times 10^{-4}) + (0.001)[(-0.315248246)] = -9.45751646 \times 10^{-4}$$

$$t_4 = t_3 + \Delta t = 0.002 + 0.001 = 0.003$$

```
% Problem 9.34
clear
clc
disp('Problem 9.34: Scott Thomas')

format long
Na = 3;
a = 5.0;
grav = 9.81;
L = 1.0;

delta_ta = 0.001;
x1a = zeros(1,Na);
x2a = zeros(1,Na);
ga = zeros(1,Na);

ta = zeros(1,Na);
x1a(1) = 0.5;
x2a(1) = 0.0;

for k = 1:Na
    x1a(k+1) = x1a(k) + x2a(k)*delta_ta;
    ga(k) = 1/L*(a*cos(x1a(k)) - grav*sin(x1a(k)));
    x2a(k+1) = x2a(k) + ga(k)*delta_ta;
    ta(k+1) = ta(k) + delta_ta;
end
ga
x1a
x2a
```

```
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ga =

  -0.315251724255368  -0.315251724255368  -0.315248254527845


x1a =

   0.500000000000000   0.500000000000000   0.499999684748276   0.499999054244827


x2a =

   1.0e-03 *

                   0  -0.315251724255368  -0.630503448510735  -0.945751703038580

>>
```

```
% Problem 9.34
clear
clc
disp('Problem 9.34: Scott Thomas')

format long
Na = 1000;
```

```
a = 5.0;
grav = 9.81;
L = 1.0;

delta_ta = 0.01;
x1a = zeros(1,Na);
x2a = zeros(1,Na);
ga = zeros(1,Na);

ta = zeros(1,Na);
x1a(1) = 0.5;
x2a(1) = 0.0;

for k = 1:Na
    x1a(k+1) = x1a(k) + x2a(k)*delta_ta;
    ga(k) = 1/L*(a*cos(x1a(k)) - grav*sin(x1a(k)));
    x2a(k+1) = x2a(k) + ga(k)*delta_ta;
    ta(k+1) = ta(k) + delta_ta;
end

Nb = 10000;
delta_tb = 0.001;
x1b = zeros(1,Nb);
x2b = zeros(1,Nb);
gb = zeros(1,Nb);

tb = zeros(1,Nb);
x1b(1) = 0.5;
x2b(1) = 0.0;

for k = 1:Nb
    x1b(k+1) = x1b(k) + x2b(k)*delta_tb;
    gb(k) = 1/L*(a*cos(x1b(k)) - grav*sin(x1b(k)));
    x2b(k+1) = x2b(k) + gb(k)*delta_tb;
    tb(k+1) = tb(k) + delta_tb;
end

Nc = 100000;
delta_tc = 0.0001;
x1c = zeros(1,Nc);
x2c = zeros(1,Nc);
gc = zeros(1,Nc);

tc = zeros(1,Nc);
x1c(1) = 0.5;
x2c(1) = 0.0;

for k = 1:Nc
    x1c(k+1) = x1c(k) + x2c(k)*delta_tc;
    gc(k) = 1/L*(a*cos(x1c(k)) - grav*sin(x1c(k)));
    x2c(k+1) = x2c(k) + gc(k)*delta_tc;
    tc(k+1) = tc(k) + delta_tc;
end

plot(ta, x1a, tb, x1b, tc, x1c)
xlabel('Time t'), ylabel('Function y(t)')
title('Problem 9.34: Scott Thomas')
legend('\Delta t = 0.01 seconds','\Delta t = 0.001 seconds','\Delta t = 0.0001 seconds','Location','NorthWest')
%axis([0 50 -3 5])
```

Problem 9.34: Scott Thomas