ME 1020 Engineering Programming with MATLAB

Problem 9.8:

8. A cone-shaped paper drinking cup (like the kind supplied at water fountains) has a radius R and a height H. If the water height in the cup is h, the water volume is given by

$$V = \frac{1}{3}\pi\left(\frac{R}{H}\right)^2 h^3$$

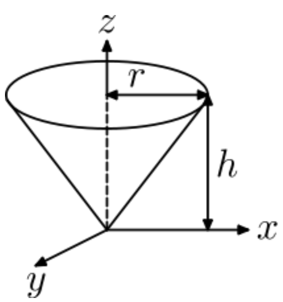
Suppose that the cup's dimensions are $R = 1.5$ in. and $H = 4$ in.

a. If the flow rate from the fountain into the cup is 2 in.^{3}/s, how long will it take to fill the cup to the brim?

b. If the flow rate from the fountain into the cup is given by $2(1 - e^{-2t})$ in.^{3}/s, how long will it take to fill the cup to the brim?

Problem setup:

Part a):



Volume within the cup as a function of time can be found by direct integration:

$$V(t) = \int_0^t \dot{V} dt + V(0)$$

where $\dot{V}$ is the volumetric flow rate of water going into the cup, and $V(0)$ is the initial volume of water within the cup. For Part a), $\dot{V} = 2.0 \text{ in}^3/\text{s}$ and $V(0) = 0$.

$$V(t) = \int_0^t 2dt = 2t$$

The volume of water in the cup is related to the height of the water in the cup:

$$V(t) = \frac{\pi}{3}\left(\frac{R}{H}\right)^2 h^3$$

The cup is full when $h = H = 4.0$ inches:

$$V = 2t = \frac{\pi}{3}\left(\frac{R}{H}\right)^2 H^3$$

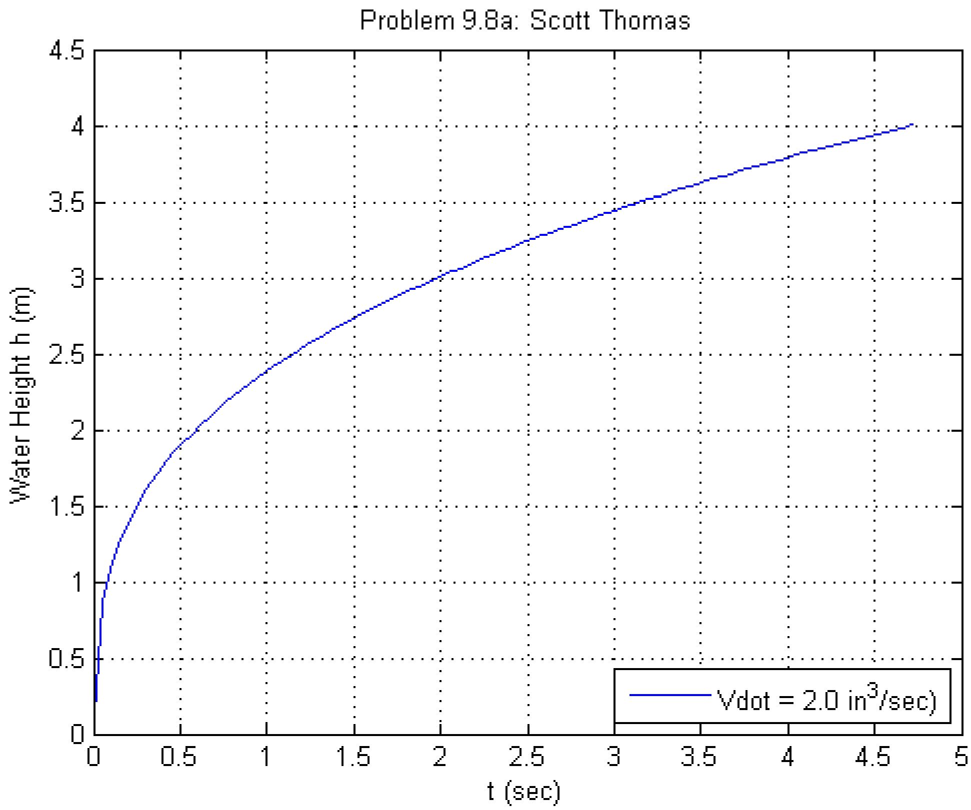
$$t = \frac{\pi}{6} R^2 H = \frac{\pi}{6}(1.5)^2(4.0) = 4.1724 \text{ seconds}$$

The height of water as a function of time is related to the volume of water in the cup.

$$h(t) = \left[\frac{3}{\pi}\left(\frac{H}{R}\right)^2 V(t)\right]^{1/3}$$

```
% Problem 9.8a
clear
clc
disp('Problem 9.8a: Scott Thomas')
R = 1.5; %in
H = 4.0; %in
disp('Part a):')
N = 101;
Vdot = 2*ones(1,N);
h = zeros(1,N);
tfinal = 0;
while h(N) < 4.0
    tfinal = tfinal + 0.001;
t = linspace(0,tfinal,N);
V = zeros(1,N);
V(1) = 0.0;
for k = 1:N-1
V(k+1) = V(k) + 0.5*(t(k+1) - t(k))*(Vdot(k) + Vdot(k+1));
end
V;
h = (3/pi*(H/R)^2*V).^(1/3);
end
h(N)
disp('Time to Fill:')
t(N)

plot(t,h), xlabel('t (sec)')
ylabel('Water Height h (m)')
title('Problem 9.8a: Scott Thomas')
grid on
legend('Vdot = 2.0 in^3/sec)', 'Location', 'SouthEast')
```

```
Problem 9.8a: Scott Thomas
Part a):

ans =

   4.0002e+00

Time to Fill:

ans =

   4.7130e+00
```



Part b):

$$V(t) = \int_0^t \dot{V} dt + V(0) = \int_0^t (1 - e^{-2t}) dt$$

From Wikipedia:

$$\int e^{cx}\, \mathrm{d}x = \frac{1}{c} e^{cx}$$

$$V(t) = \left[t - \frac{1}{(-2)} e^{-2t}\right]_0^t$$

$$V(t) = \left[t + \frac{1}{2} e^{-2t}\right] - \left[(0) + \frac{1}{2} e^{-2\cdot 0}\right]$$

$$V(t) = t + \frac{1}{2} e^{-2t} - \frac{1}{2}$$

$$V(t) = t + \frac{1}{2}(e^{-2t} - 1)$$

This equation cannot be solved for t directly. This is called a transcendental equation. The volume of water when the cup is full is given by:

$$V = \frac{\pi}{3}\left(\frac{R}{H}\right)^2 h^3 = \frac{\pi}{3}\left(\frac{R}{H}\right)^2 H^3 = \frac{\pi}{3}R^2 H = \frac{\pi}{3}(1.5)^2(4.0) = 9.4248\ \text{in}^3$$

Change the final time in the MATLAB program below until $V = 9.4248\ \text{in}^3$ or $h = 4.0$ in.

```
% Problem 9.8b
clear
clc
disp('Problem 9.8b: Scott Thomas')

R = 1.5; %in
H = 4.0; %in

disp('Part b):')
N = 101;
% Iterate on the final time to find when V = 9.4248 in^3 or h = 4.0 in
timefinal = 5.2124;
t = linspace(0,timefinal,N);
Vdot = 2*(1 - exp(-2*t));
%h = zeros(1,N);
V = zeros(1,N);
V(1) = 0.0;
for k = 1:N-1
V(k+1) = V(k) + 0.5*(t(k+1) - t(k))*(Vdot(k) + Vdot(k+1));
end
V;
h = (3/pi*(H/R)^2*V).^(1/3);
disp('V(N) =')
V(N)
disp('h(N) =')
h(N)
disp('Time to Fill:')
t(N)
plot(t,h), xlabel('t (sec)')
ylabel('Water Height, h (in)')
title('Problem 9.8b: Scott Thomas')
grid on
legend('Vdot =  2(1 - e^{-2t} ) (in^3/sec)','Location', 'SouthEast')
```

```
Problem 9.8b: Scott Thomas
Part b):
V(N) =

ans =

   9.4239e+00

h(N) =

ans =

   3.9999e+00

Time to Fill:

ans =

   5.2124e+00
```