

USING GAUSSIANS IN IMAGE FILTERING AND DATA APPROXIMATION

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Abstract. Gaussians are widely used in image filtering because of their smooth spectrum and localized spatial characteristics. In this paper, data approximation is viewed as a filtering operation, and by studying the frequency characteristics of Gaussians as filters, their behaviors in data approximation are predicted. Frequency characteristics of Gaussian basis functions are compared with those of multiquadrics and inverse multiquadrics to determine the reproducibility of variations in data by these functions.

INTRODUCTION

Given points $\{\mathbf{x}_i : i = 1, \dots, N\}$ in \mathcal{R}^n with associated values $\{F_i : i = 1, \dots, N\}$, a function f that approximates the values can be defined by

$$f(\mathbf{x}_i) \approx F_i, \quad i = 1, \dots, N. \quad (1)$$

If radial basis functions are used, f can be defined by

$$f(\mathbf{x}) = \sum_{i=1}^N \alpha F_i b_i(\mathbf{x}). \quad (2)$$

In this paper, it is assumed that data are uniformly spaced and $n = 1$. Therefore, formula (2) can be written as

$$f(x) = \sum_{i=1}^N \alpha F_i b_i(x). \quad (3)$$

α is a scaling factor determined such that $\alpha = 1/\sum_i b_i(x_j)$. Formula (3) defines the value at x as a weighted sum of basis functions centered at the data points and evaluated at x , the weights being the data values F_i scaled by α . Formula (3) can also be considered a filtering operation with F_i 's representing samples from a signal, $\alpha b_i(x)$ representing the filter where now $\alpha = 1/\sum_j b_i(x_j)$, and $f(x)$ representing an approximation to the filtered signal. We assume that given data are uniformly spaced and periodic so that $\sum_j b_i(x_j) = \sum_i b_i(x_j)$.

Frequency characteristics of Gaussians [6] [11], multiquadrics (MQ) [7] [8], and inverse multiquadrics (IMQ) [4] [7] defined, respectively, by

$$b_i(x) = e^{-(x-x_i)^2/2\sigma^2}, \quad (4)$$

$$b_i(x) = [(x-x_i)^2 + r^2]^{\frac{1}{2}}, \quad (5)$$

$$b_i(x) = [(x-x_i)^2 + r^2]^{-\frac{1}{2}} \quad (6)$$

will be studied. Since approximation and filtering are equivalent, by studying the behaviors of these basis functions in filtering, their behaviors in approximation will be determined.

DESIRED PROPERTIES OF SMOOTHING FILTERS

There are two factors that determine the usefulness of a smoothing filter [10]. The first reduces the range of resolutions over which variations in output appear by requiring that the filter's variance, $\Delta\omega$, in the frequency domain be small. The second factor increases the spatial localization of the filter by requiring that

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the contribution to each point in the filtered data to arise from a smooth average in a small neighborhood. Therefore, a desirable filter should also be smooth and localized in the spatial domain, having a small spatial variance, Δx .

These localization requirements in the spatial and frequency domains, however, are conflicting and related by the uncertainty principle, which states that $\Delta x \Delta \omega \geq \frac{1}{4\pi}$. It has been shown that Gaussians are the only functions that provide the optimal trade-off between these conflicting requirements with $\Delta x \Delta \omega = \frac{1}{4\pi}$ [3, pp. 160–163].

In data approximation, localization in the spatial domain refers to the ability of a basis function to allow points that are closer to x to contribute more to the shape of f in the neighborhood of x . Spatial localization makes a function locally sensitive to data, and when a new data point is added to or removed from a data set, the shape of the function will change more as one moves closer to the point. When an error tolerance is allowed, it becomes possible to compute $f(x)$ using data in a restricted domain around x . The influence of points farther away will be negligibly small and not measurable with the required tolerance. By examining the shapes of Gaussian, MQ, and IMQ basis functions from formulas (4)–(6), we see that Gaussians and IMQ are locally sensitive and can be made more spatially localized by decreasing parameters σ and r . However, MQ are not locally sensitive and addition or deletion of a data point to a data set requires use of the entire data to compute $f(x)$.

In data approximation, localization in the frequency domain refers to the ability of a function to reconstruct a desired range of spatial frequencies in a given data set. To study the frequency characteristics of a basis function, the Fourier transforms of the basis function are used. The Fourier transform of a function defined by formula (3) is characterized by the Fourier transforms of its basis functions. To find the effectiveness of these basis functions in reproducing spatial frequencies in a data set, three sets of data as shown in Figs. 1a-c were generated. Figure 1a shows uniformly-spaced samples from a signal that contains a wide range of frequencies, Fig. 1b shows samples from a signal that contains either very high or very low frequencies, and Fig. 1c shows samples from a signal that contains very low frequencies.

The spatial frequency contents of a signal can be determined by computing its Fourier transform. We would like to determine the basis function which can best reconstruct the spatial frequencies in a signal. Therefore, for fixed values of σ and r , the Fourier transforms of the basis functions were computed and the root-mean-squared error (RMSE) between them and the Fourier transforms of the signals were determined. The Fourier transform coefficients were normalized in such a way that the magnitude of the zeroth order coefficient became equal to one. Parameters σ and r of the filters were iteratively varied until minima were reached in the errors. The optimal values of σ and r in these experiments and the minimum RMSE for the three filters and the three data sets were determined and summarized in Table I. From these experiments we see that Gaussians can represent highly varying data better than MQ and IMQ, while IMQ can represent very smooth data better than Gaussians and MQ, and when data contain variations at a wide range of resolutions, MQ can represent them better than Gaussians and IMQ. Buhmann and Powell [2] have studied the localization properties of MQ and IMQ arriving at similar conclusions.

UNIQUENESS OF GAUSSIANS

In data approximation, it is desirable to produce functions that have the least fluctuations. Fluctuations in a function can be characterized by the number of its inflection points. If data are approximated using Gaussian basis functions, increasing the standard deviations of Gaussians will yield smoother functions with fewer inflections. It has been shown that Gaussians are the only functions whose local maxima of the approximating function always increase and whose local minima always decrease as σ is increased, and inflection points observed at a small σ will vanish with increasing σ [1].

Gaussians have the unique ability to describe data from various physical phenomena. If we assume a given set of data represents initial heat distribution at time zero in a rod, the heat distribution after time σ can be described by a sum of Gaussians of standard deviation σ [3, pp. 355] [9].

Gaussians are the result of the central limit theorem [5, p. 197, p. 203]: Assuming k integers are selected at random without replacement from the first K integers 1 to K , and denoting the i th selected integer by X_i and letting

$$Y_k = X_1 + X_2 + \cdots + X_k, \quad (7)$$

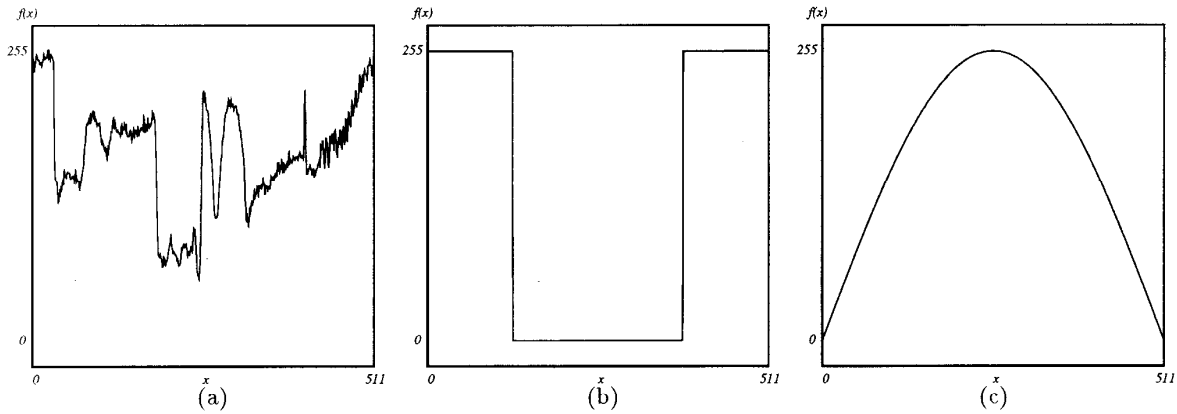


Fig. 1. Three data sets containing 512 uniformly-spaced values.

the distribution of Y_k approaches a Gaussian centered at μ_k and standard deviation σ_k , where μ_k and σ_k^2 represent the mean and variance of Y_k , respectively.

CONCLUSIONS

In this paper the spatial frequency characteristics of Gaussians, MQ, and IMQ were examined. Gaussians are most suitable for approximating highly varying data, IMQ best approximate smoothly varying data, and when a data set contains smoothly as well as highly varying components, MQ perform the best. MQ are, however, not spatially localized, meaning that changing a single data point will change the entire output. Gaussians and IMQ, on the other hand, are spatially localized and changing a data point will change the output mostly in the neighborhood of the point, and the effect vanishes as one moves away from the point.

A nice property of Gaussians in data approximation that is not shared by other basis functions is that the user can monotonically decrease the number of inflection points in the approximating function by increasing the standard deviation of the Gaussians.

Table I. Root-mean-square errors of optimal Gaussians, MQ, and IMQ reproducing spatial frequencies in data sets of Fig. 1.

	Gaussian		MQ		IMQ	
	RMSE	σ	RMSE	r	RMSE	r
Fig. 1a	4314e-7	231.37	4291e-7	181.33	4453e-7	178.91
Fig. 1b	8972e-7	87.08	9722e-7	11.09	9922e-7	28.13
Fig. 1c	1598e-7	132.48	1297e-7	48.12	896e-7	62.59

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