APPENDIX F

Some Probability Densities and Distributions

For convenience of reference we list below the probability density $f_X(x)$ and distribution function $F_X(x)$ for some well-known distributions. Where appropriate, we also give the mean $\bar{X}$, variance σ_X^2, and characteristic function $\Phi_X(\omega)$.

A number of constants and functions are used as defined below:†

$$a, a_1, a_2, b, b_1, b_2, \sigma, \text{ and } p \text{ are real constants} \tag{F-1a}$$

$$N \text{ is a positive integer} \tag{F-1b}$$

$$\delta(\xi) = \text{impulse function of (2.3-2)} \tag{F-1c}$$

$$u(\xi) = \text{unit-step function of (2.2-4)} \tag{F-1d}$$

$$\text{rect}\,(\xi) = \text{rectangular function of (E-2)} \tag{F-1e}$$

$$\begin{aligned}\Gamma(x) &= \int_0^\infty \xi^{x-1} e^{-\xi}\, d\xi \qquad \text{Re}\,(x) > 0 \\ &= \text{gamma function}\end{aligned} \tag{F-1f}$$

$$\begin{aligned}P(\alpha, \beta) &= \frac{1}{\Gamma(\alpha)} \int_0^\beta \xi^{\alpha-1} e^{-\xi}\, d\xi \qquad \text{Re}\,(\alpha) > 0 \\ &= \text{incomplete gamma function}\end{aligned} \tag{F-1g}$$

$$P(x|N) = \frac{1}{2^{N/2}\Gamma(N/2)}\int_0^x \xi^{(N/2)-1}e^{-\xi/2}\,d\xi$$
$$= \text{chi-square probability function}$$
$$= P\left(\frac{N}{2}, \frac{x}{2}\right) \tag{F-1h}$$

$$I(u,p) = \frac{1}{\Gamma(p+1)}\int_0^{u\sqrt{p+1}} \xi^p e^{-\xi}\,d\xi$$
$$= \text{Pearson's form of incomplete gamma function (Pearson, 1934)}$$
$$= P(p+1, u\sqrt{p+1}) \tag{F-1i}$$

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x \xi^{a-1}(1-\xi)^{b-1}\,d\xi$$
$$= \text{incomplete beta function} \tag{F-1j}$$

$$F(x) = \text{gaussian distribution of (B-3)} \tag{F-1k}$$

$$I_n(x) = (x/2)^n \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!(n+k)!}$$
$$= \frac{1}{\pi}\int_0^{\pi} e^{x\cos(\theta)}\cos(n\theta)\,d\theta$$
$$= \text{modified Bessel function of first kind of order } n = 0, 1, 2, \ldots \tag{F-1l}$$

$$Q(\alpha, \beta) = \int_{\beta}^{\infty} \xi I_0(\alpha\xi)\exp\left[\frac{-(\xi^2+\alpha^2)}{2}\right]d\xi \tag{F-1m}$$

The functions of (F-1*f*) through (F-1*j*) and that of (F-1*l*) are discussed in detail in Abramowitz and Stegun, editors (1964). $Q(\alpha, \beta)$ is Marcum's *Q*-function; it is tabulated in Marcum (1950).

DISCRETE FUNCTIONS

Bernoulli

For $0 < p < 1$

$$f_X(x) = (1-p)\delta(x) + p\delta(x-1) \tag{F-2}$$
$$F_X(x) = (1-p)u(x) + pu(x-1) \tag{F-3}$$
$$\bar{X} = p \tag{F-4}$$
$$\sigma_X^2 = p(1-p) \tag{F-5}$$
$$\Phi_X(\omega) = 1 - p + pe^{j\omega} \tag{F-6}$$

Binomial

For $0 < p < 1$ and $N = 1, 2, \dots$

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k) \tag{F-7}$$

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k) \tag{F-8}$$

$$\bar{X} = Np \tag{F-9}$$

$$\sigma_X^2 = Np(1-p) \tag{F-10}$$

$$\Phi_X(\omega) = [1 - p + pe^{j\omega}]^N \tag{F-11}$$

Pascal†

For $0 < p < 1$ and $N = 1, 2, \dots$

$$f_X(x) = \sum_{k=N}^{\infty} \binom{k-1}{N-1} p^N (1-p)^{k-N} \delta(x-k) \tag{F-12}$$

$$F_X(x) = \sum_{k=N}^{\infty} \binom{k-1}{N-1} p^N (1-p)^{k-N} u(x-k) \tag{F-13}$$

$$\bar{X} = \frac{N}{p} \tag{F-14}$$

$$\sigma_X^2 = \frac{N(1-p)}{p^2} \tag{F-15}$$

$$\Phi_X(\omega) = p^N e^{jN\omega} [1 - (1-p)e^{j\omega}]^{-N} \tag{F-16}$$

Poisson

For $b > 0$

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k) \tag{F-17}$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k) \tag{F-18}$$

$$\bar{X} = b \tag{F-19}$$

$$\sigma_X^2 = b \tag{F-20}$$

$$\Phi_X(\omega) = \exp[b(e^{j\omega} - 1)] \tag{F-21}$$

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†Blaise Pascal (1623–1662) was a French mathematician.

CONTINUOUS FUNCTIONS

Arcsine

For $a > 0$

$$f_X(x) = \frac{\text{rect}\,(x/2a)}{\pi\sqrt{a^2 - x^2}} \tag{F-22}$$

$$F_X(x) = \begin{cases} 0 & -\infty < x < -a \\ \dfrac{1}{2} + \dfrac{1}{\pi}\sin^{-1}\left(\dfrac{x}{a}\right) & -a \le x < a \\ 1 & a \le x < \infty \end{cases} \tag{F-23}$$

$$\bar{X} = 0 \tag{F-24}$$

$$\sigma_X^2 = \frac{a^2}{2} \tag{F-25}$$

Beta

For $a > 0$ and $b > 0$

$$f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}[u(x) - u(x-1)]x^{a-1}(1-x)^{b-1} \tag{F-26}$$

$$F_X(x) = \begin{cases} I_x(a,b)u(x) & x < 1 \\ 1 & x \ge 1 \end{cases} \tag{F-27}$$

$$\bar{X} = \frac{a}{a+b} \tag{F-28}$$

$$\sigma_X^2 = \frac{ab}{(a+b)^2(a+b+1)} \tag{F-29}$$

Cauchy

For $b > 0$ and $-\infty < a < \infty$

$$f_X(x) = \frac{(b/\pi)}{b^2 + (x-a)^2} \tag{F-30}$$

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{b}\right) \tag{F-31}$$

$$\bar{X} = \text{is undefined} \tag{F-32}$$

$$\sigma_X^2 = \text{is undefined} \tag{F-33}$$

$$\Phi_X(\omega) = e^{ja\omega - b|\omega|} \tag{F-34}$$

Chi-Square with N Degrees of Freedom

For $N = 1, 2, \ldots$

$$f_X(x) = \frac{x^{(N/2)-1}}{2^{N/2}\Gamma(N/2)} e^{-x/2} u(x) \tag{F-35}$$

$$F_X(x) = P(x|N) = P\left(\frac{N}{2}, \frac{x}{2}\right) \tag{F-36}$$

$$\bar{X} = N \tag{F-37}$$

$$\sigma_X^2 = 2N \tag{F-38}$$

$$\Phi_X(\omega) = (1 - j2\omega)^{-N/2} \tag{F-39}$$

Erlang

For $N = 1, 2, \ldots$ and $a > 0$

$$f_X(x) = \frac{a^N x^{N-1} e^{-ax}}{(N-1)!} u(x) \tag{F-40}$$

$$F_X(x) = \left[1 - e^{-ax} \sum_{n=0}^{N-1} \frac{(ax)^n}{n!}\right] u(x) \tag{F-41}$$

$$\bar{X} = \frac{N}{a} \tag{F-42}$$

$$\sigma_X^2 = \frac{N}{a^2} \tag{F-43}$$

$$\Phi_X(\omega) = \left(\frac{a}{a - j\omega}\right)^N \tag{F-44}$$

Exponential

For $a > 0$

$$f_X(x) = a e^{-ax} u(x) \tag{F-45}$$

$$F_X(x) = [1 - e^{-ax}] u(x) \tag{F-46}$$

$$\bar{X} = \frac{1}{a} \tag{F-47}$$

$$\sigma_X^2 = \frac{1}{a^2} \tag{F-48}$$

$$\Phi_X(\omega) = \frac{a}{a - j\omega} \tag{F-49}$$

Gamma

For $a > 0$ and $b > 0$

$$f_X(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)} u(x) \tag{F-50}$$

$$F_X(x) = I\left(\frac{ax}{\sqrt{b}}, b-1\right) u(x) \tag{F-51}$$

$$\bar{X} = \frac{b}{a} \tag{F-52}$$

$$\sigma_X^2 = \frac{b}{a^2} \tag{F-53}$$

$$\Phi_X(\omega) = \left(\frac{a}{a - j\omega}\right)^b \tag{F-54}$$

Note that if b is a positive integer the gamma density becomes the Erlang density. Also if $b = N/2$, for $N = 1, 2, \ldots,$ and $a = \frac{1}{2}$ the gamma density becomes the chi-square density.

Gaussian-Univariate

For $b > 0$ and $-\infty < a < \infty$

$$f_X(x) = (\pi b)^{-1/2} e^{-(x-a)^2/b} \tag{F-55}$$

$$F_X(x) = F\left(\frac{x-a}{\sqrt{b/2}}\right) \tag{F-56}$$

$$\bar{X} = a \tag{F-57}$$

$$\sigma_X^2 = \frac{b}{2} \tag{F-58}$$

$$\Phi_X(\omega) = e^{j\omega a - (\omega^2 b/4)} \tag{F-59}$$

Gaussian-Bivariate

For $-\infty < a_1 < \infty$, $-\infty < a_2 < \infty$, $b_1 > 0$, $b_2 > 0$ and $-1 \le \rho \le 1$

$$\begin{aligned} f_{X_1, X_2}(x_1, x_2) = {} & [\pi^2 b_1 b_2 (1 - \rho^2)]^{-1/2} \\ & \cdot \exp\left\{\frac{-1}{(1-\rho^2)}\left[\frac{(x_1 - a_1)^2}{b_1}\right.\right. \\ & \left.\left. - \frac{2\rho(x_1 - a_1)(x_2 - a_2)}{\sqrt{b_1 b_2}} + \frac{(x_2 - a_2)^2}{b_2}\right]\right\} \end{aligned} \tag{F-60}$$

$$F_{X_1,X_2}(x_1, x_2) = L\left(-\left[\frac{x_1 - a_1}{\sqrt{b_1/2}}\right], -\left[\frac{x_2 - a_2}{\sqrt{b_2/2}}\right], \rho\right) \qquad \text{(F-61)}$$

where $L(x_1, x_2, \rho)$ is a probability function discussed extensively and graphed in Abramowitz and Stegun, editors (1964), p. 936. Also

$$\bar{X}_1 = a_1 \qquad \text{(F-62)}$$

$$\bar{X}_2 = a_2 \qquad \text{(F-63)}$$

$$\sigma_{X_1}^2 = b_1/2 \qquad \text{(F-64)}$$

$$\sigma_{X_2}^2 = b_2/2 \qquad \text{(F-65)}$$

$$\Phi_{X_1,X_2}(\omega_1, \omega_2) = \exp\{j\omega_1 a_1 + j\omega_2 a_2 - \tfrac{1}{4}[\omega_1^2 b_1 + 2\rho\omega_1\omega_2\sqrt{b_1 b_2} + \omega_2^2 b_2]\} \qquad \text{(F-66)}$$

Laplace

For $b > 0$ and $-\infty < a < \infty$

$$f_X(x) = \frac{b}{2} e^{-b|x-a|} \qquad \text{(F-67)}$$

$$F_X(x) = \begin{cases} \frac{1}{2} e^{b(x-a)} & -\infty < x < a \\ 1 - \frac{1}{2} e^{-b(x-a)} & a \le x < \infty \end{cases} \qquad \text{(F-68)}$$

$$\bar{X} = a \qquad \text{(F-69)}$$

$$\sigma_X^2 = \frac{2}{b^2} \qquad \text{(F-70)}$$

$$\Phi_X(\omega) = b^2 \frac{e^{ja\omega}}{b^2 + \omega^2} \qquad \text{(F-71)}$$

Log-Normal

For $-\infty < a < \infty$, $-\infty < b < \infty$, and $\sigma > 0$

$$f_X(x) = \frac{u(x-b) e^{-[\ln(x-b)-a]^2/2\sigma^2}}{\sqrt{2\pi}(x-b)\sigma} \qquad \text{(F-72)}$$

$$F_X(x) = u(x-b) F\{\sigma^{-1}[\ln(x-b) - a]\} \qquad \text{(F-73)}$$

$$\bar{X} = b + \exp\left(a + \frac{\sigma^2}{2}\right) \qquad \text{(F-74)}$$

$$\sigma_X^2 = [\exp(\sigma^2) - 1] \exp(2a + \sigma^2) \qquad \text{(F-75)}$$

Rayleigh

For $-\infty < a < \infty$ and $b > 0$

$$f_X(x) = \frac{2}{b}(x-a)e^{-(x-a)^2/b}u(x-a) \tag{F-76}$$

$$F_X(x) = [1 - e^{-(x-a)^2/b}]u(x-a) \tag{F-77}$$

$$\bar{X} = a + \sqrt{\frac{\pi b}{4}} \tag{F-78}$$

$$\sigma_X^2 = \frac{b(4-\pi)}{4} \tag{F-79}$$

Rice [Thomas (1969), Middleton (1960)]

For $a > 0$ and $b > 0$

$$f_X(x) = \frac{x}{b^2}e^{-(a^2+x^2)/2b^2} I_0\left(\frac{ax}{b^2}\right)u(x) \tag{F-80}$$

$$F_X(x) = \left[1 - Q\left(\frac{a}{b}, \frac{x}{b}\right)\right]u(x) \tag{F-81}$$

$$\bar{X} = b\sqrt{\frac{\pi}{2}}e^{-k^2/4}\left[\left(1+\frac{k^2}{2}\right)I_0\left(\frac{k^2}{4}\right) + \frac{k^2}{2}I_1\left(\frac{k^2}{4}\right)\right] \tag{F-82}$$

$$\sigma_X^2 = b^2(2+k^2) - (\bar{X})^2 \tag{F-83}$$

$$k^2 = \frac{a^2}{b^2} \tag{F-84}$$

Uniform

For $-\infty < a < b < \infty$

$$f_X(x) = \frac{u(x-a) - u(x-b)}{b-a} \tag{F-85}$$

$$F_X(x) = \begin{cases} \dfrac{(x-a)u(x-a)}{b-a} & x < b \\ 1 & x \geq b \end{cases} \tag{F-86}$$

$$\bar{X} = \frac{a+b}{2} \tag{F-87}$$

$$\sigma_X^2 = \frac{(b-a)^2}{12} \tag{F-88}$$

$$\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)} \tag{F-89}$$

Weibull

For $a > 0$ and $b > 0$

$$f_X(x) = abx^{b-1}e^{-ax^b}u(x) \tag{F-90}$$

$$F_X(x) = [1 - e^{-ax^b}]u(x) \tag{F-91}$$

$$\bar{X} = \frac{\Gamma(1 + b^{-1})}{a^{1/b}} \tag{F-92}$$

$$\sigma_X^2 = \frac{\Gamma(1 + 2b^{-1}) - [\Gamma(1 + b^{-1})]^2}{a^{2/b}} \tag{F-93}$$

Note that if $b = 2$ the Weibull density becomes a Rayleigh density.