MID-TERM

1 hr. 30 min. 2*nd* May, 2001

The Exam will be graded out of 30 marks. You may solve all the problems to earn extra points.

1. In a binary (0/1) transmission system, it was found that $P(1) = 0.55$. You designed a transmission line system for your company that has Probability of error/bit = 0.0011. A customer will not buy this product if there are more than 1 error out of 500 bits. In order to test the product you designed, the customer sends 500 known test bits through your system. What is the probability that your company will not win this customer's contract? (4 marks)

2. In Laboratory simulation using a test binary communication channel, it was found that 99% of the 1's and 98% of 0's transmitted are received without error. This binary communication channel is to be used in a system that transmits 1 and 0 with probabilities P(1) = 0.54 = 1 - P(0).
   a. Find the probability of receiving a 1.
   b. Given that a 0 is received, find the probability that a 1 was sent.
   c. If $\mathbf{X}$ and $\mathbf{Y}$ denote the binary random variables at the transmission and receiving ends, respectively, find the mean and variances of $\mathbf{X}$ and $\mathbf{Y}$. (5 marks)

3. A random variable $\mathbf{X}$ is uniformly distributed between 1 and 2 and another random variable $\mathbf{Y}$ is uniformly distributed between -2 and 0. Find the probability density function of,

$$\mathbf{Z} = 2\mathbf{X} + \frac{1}{2}\mathbf{Y}^2.$$

HINT : First find the density functions of $2\mathbf{X}$ and $\frac{1}{2}\mathbf{Y}^2$ separately.
You may need : $\frac{d}{dx}\ (x^{-\frac{1}{2}}) = 2\sqrt{x}.$ (10 marks)

4. The radial distance to the impact points for shells fired on land by a cannon can be approximated by a gaussian distribution with mean = 3000 feet and standard deviation = 100 feet when the cannon is aimed at a target located at 2900 feet distance from the cannon.
   a. Find the probability that the shells fired will fall within (i) ± 150 feet and (ii) ± 20 feet of the target.
   b. Find the probability that the shells will fall at 3200 feet or above. (5 marks)

5. Find the mean and the auto-covariance of the random process defined as $\mathbf{X}(t) = 2e^{-At}\sin(wt + \phi)$, where, the random variables A and ϕ are independent. A is uniformly distributed between 2 and 4 and ϕ is uniformly distributed between $-\pi$ and π. Plot a few realizations of the ensemble. (8 marks)

Total Marks 32

EXAMINATION BOOK

(1)

PLEASE USE BOTH SIDES OF PAPER

Name: MID-TERM SOLUTIONS
Course: EE761
Instructor: SHAW
Date: MAY 2, 2001

1. $P(\text{error}) = 0.0011 = P(e)$
   $P(\text{no error}) = 0.9989 = P(\bar{e})$

NOTE: THE CUSTOMER WILL NOT BUY IF THERE ARE 2 OR MORE ERRORS.

$\therefore\ P(\text{NO CONTRACT}) = 1 - [P(0\ \text{error}) + P(1\ \text{error})]$

OUT OF 500 BITS

(a) 0 ERRORS CAN OCCUR WITH probability $= (P(e))^0 (P(\bar{e}))^{500}$

$$= (0.9989)^{500} = 0.5768$$

(b) 1 ERROR CAN OCCUR WITH PROBABILITY

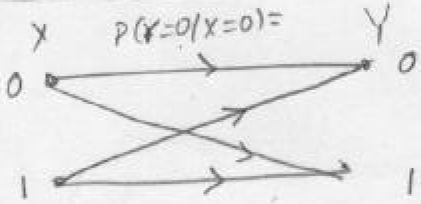
$$\binom{500}{1} (P(e))^1 (P(\bar{e}))^{499}$$

$$= \frac{500!}{(500-1)!\,1!} (0.0011)(0.9989)^{499}$$

$$= (500)(0.0011)(0.9989)^{499} = 0.3176$$

$\therefore\ P(\text{NO Contract}) = 1 - (0.9989)^{500} - (0.55)(0.9989)^{499} = 1 - (0.5768 + 0.3176) = 0.10$

NOTE: THE GIVEN FACT THAT $P(1) = 0.55 = 1 - P(0)$ IS IMMATER FOR THIS SCENARIO, AS ERROR CAN AFFECT BOTH 0^s AND IS EQUALLY

2. 

GIVEN, $P(X=1) = 0.54 \Rightarrow P(X=0) = 1 - 0.5$
$= 0.46$

ALSO GIVEN,

$P(Y=0|X=0) = 0.98 \Rightarrow P(Y=1|X=0) = 0.02$

& $P(Y=1|X=1) = 0.99 \Rightarrow P(Y=0|X=1) = 0.01$

a. $P(Y=1) = P(X=1, Y=1) + P(X=0, Y=1)$

$$= P(Y=1|X=1)P(X=1) + P(Y=1|X=0)P(X=0)$$

$$= (0.99)(0.54) + (0.02)(0.46) = 0.5438$$

$\therefore\ P(Y=0) = 1 - 0.5438 = 0.4562$

(b) $P(X=1|Y=0) = \dfrac{P(X=1, Y=0)}{P(Y=0)} = \dfrac{P(Y=0|X=1)P(X=1)}{P(Y=0)}$ $\longrightarrow$ FROM PAR (2

$$= \frac{(0.01)(0.54)}{(0.4562)} = 0.01184$$

(c) $E[X] = (0)(P(X=0)) + (1)(P(X=1)) = 0.54$

$E[X^2] = (0^2)P(X=0) + 1^2\, P(X=1) = 0.54$

$\sigma_X^2 = E[X^2] - E^2[X] = 0.54 - (0.54)^2 = 0.2484$

$E[Y] = 0 \cdot P(Y=0) + (1)\, P(Y=1) = 0.5438$

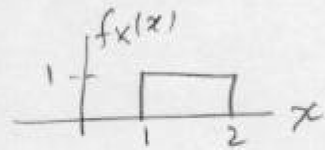
$E[Y^2] = 0.5438$ ALSO

$\sigma_Y^2 = E[Y^2] - E^2[Y] = 0.5438 - (0.5438)^2 = 0.24808$

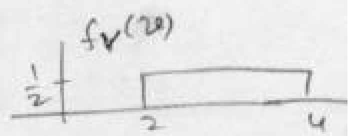
3. LET, $V = 2X$ $\quad \dfrac{dv}{dx} = 2 \quad x = \dfrac{v}{2}$

$$\therefore f_V(v) = \frac{f_X(\frac{v}{2})}{2} = 0.5\, f_X\left(\frac{v}{2}\right)$$

Given, $f_X(x) = U(1,2) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{Elsewhere} \end{cases}$



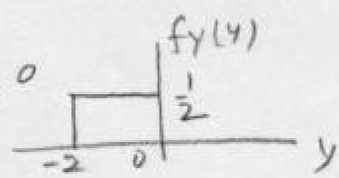
HENCE, $f_V(v) = U(2,4) = \begin{cases} \frac{1}{2} & 2 \le v \le 4 \\ 0 & \text{EISEWHER} \end{cases}$



ALSO, LET, $W = \frac{1}{2} Y^2 \Rightarrow \dfrac{dw}{dy} = y = \pm\sqrt{2w}$

$y = \pm\sqrt{2w} \rightarrow 2$ ROOTS

GIVEN, $f_Y(y) = U(-2,0) = \begin{cases} \frac{1}{2} & -2 \le y \le 0 \\ 0 & EW \end{cases}$



min of $w = 0$, when $y = 0$

max of $w = \frac{1}{2}(-2)^2 = 2$; when $y = -2$

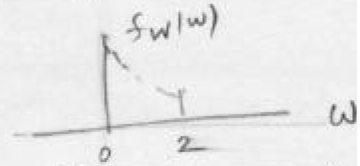
$\therefore$ W VARIES BETWEEN 0 & 2.

③

$$f_W(w) = \frac{f_Y(\sqrt{2w})}{|\sqrt{2w}|} + \frac{f_Y(-\sqrt{2w})}{|-\sqrt{2w}|}$$

$$= 0 + \frac{\frac{1}{2}}{\sqrt{2w}} = \frac{1}{2\sqrt{2w}} \quad ; \quad 0 < w \le 2$$

since Y is ALWAYS NEGATIVE



note, at $Y=0$, $W=0$ also

CHECK; $\int_{-\infty}^{\infty} f_W(w)\,dw \;\frac{\text{MUST}}{\text{BE}}\; 1$

$$= \int_0^2 \frac{1}{2\sqrt{2w}}\,dw = \frac{1}{2\sqrt{2}} 2w^{\frac{1}{2}}\Big|_0^2 = \frac{2(\sqrt{2}-0)}{2\sqrt{2}} = 1$$

CHECKS!

$$Z = 2X + \tfrac{1}{2}Y^2 \triangleq V + W \qquad \text{(BY OUR DEFINITION)}$$

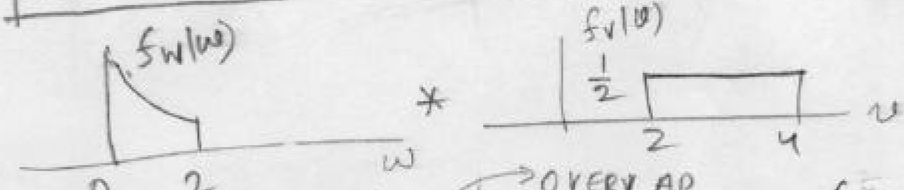
SINCE X & Y ARE INDEPENDENT, SO ARE V & W

$\therefore f_Z(z) = f_V(v) * f_W(w)$

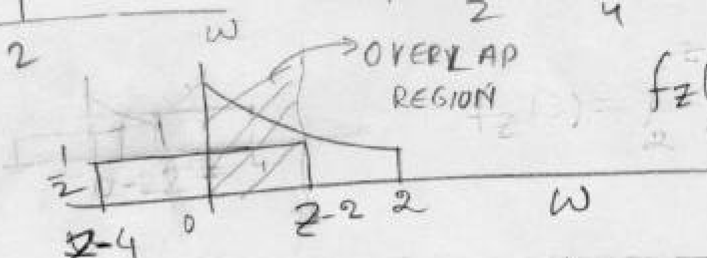
NOTE, $\min(z) = \min(v) + \min(w) = 2+0 = 2$ (BECAUSE V & W ARE POSITIVE)

$\max(z) = \max(v) + \max(w) = 4+2 = 6$

CASE I $\therefore$ $\boxed{f_Z(z) = 0}$ FOR, $z \ge 6$ & $z \le 2$



CASE II WHEN $2 \le z \le 4$,



$$f_Z(z) = \int_0^{z-2} \left(\tfrac{1}{2}\right) \frac{1}{2\sqrt{2w}}\,dw$$

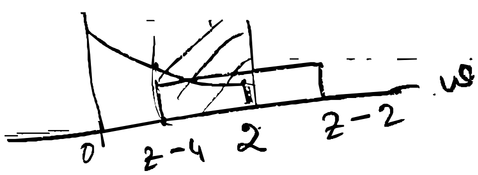
$$= \frac{1}{4\sqrt{2}}\, 2\sqrt{w}\,\Big|_0^{z-2}$$

$$\boxed{f_Z(z) = \frac{1}{2\sqrt{2}}\left(\sqrt{z-2}\right);} \quad 2 \le z \le 4$$

CASE III

WHEN $4 \le z \le 6$

overlap region



$$f_Z(z) = \int_{z-4}^{2} \left(\frac{1}{2}\right)\frac{1}{2\sqrt{2w}}\,dw = \frac{1}{4\sqrt{2}}\, 2\sqrt{w}\Big|_{z-4}^{2}$$

$$= \boxed{\frac{1}{2\sqrt{2}}\left(\sqrt{2}-\sqrt{z-4}\right)}\ ; 4 \le z \le 6$$

CHECK $\int_{-\infty}^{\infty} f_Z(z)\,dz \overset{\text{MUST}}{\underset{\text{BE}}{=}} 1 = \int_2^4 \frac{1}{2\sqrt{2}}\sqrt{z-2}\,dz + \int_4^6 \frac{1}{2\sqrt{2}}\left(\sqrt{2}-\sqrt{z-4}\right)dz$

$\int_2^4 \sqrt{z-2}\,dz$ let, $z-2=u$, $dz=du$, @ $z=2, u=0$, @ $z=4, u=2$

$$\therefore = \int_0^2 \sqrt{u}\,du = \frac{1}{1+\frac{1}{2}}\, u^{3/2}\Big|_0^2 = \frac{2}{3}\left(2^{3/2}\right)$$

similarly, FOR $\int_4^6 \sqrt{z-4}\,dz$, LET, $z-4=u \Rightarrow dz=du$, @ $z=6$, $u=2$ & @ $z=4$, $u=0$

$$= \int_0^2 \sqrt{u}\,du = \frac{2}{3}\left(2^{3/2}\right) \therefore$$

ALSO, $\int_4^6 \frac{\sqrt{2}}{2\sqrt{2}}\,dz = \frac{1}{2}\int_4^6 dz = \frac{2}{2} = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f_Z(z) = \frac{2}{3}\,2^{3/2} + 1 - \frac{2}{3}\,2^{3/2} = 1$$

CHECKS ✓

4. a. (i) $P(2900-150 \leq SHELL \leq 2900+150)$ 5

$= P(2750 \leq SHELL \leq 3050)$ BEFORE NORMALIZATION

$= P\left(\frac{2750-3000}{100} \leq S \leq \frac{3050-3000}{100}\right)$ (AFTER NORMALIZ

$= P(-2.5 \leq S \leq 0.5)$

$= F_S(0.5) - F_S(-2.5)$

$= 1 - Q(0.5) - Q(2.5)$

$= 1 - 0.3085 - 0.0062 = 0.6853$

(ii) $P(2900-20 \leq shell \leq 2900+20)$

$= P\left(\frac{2880-3000}{100} \leq S \leq \frac{2920-3000}{100}\right) =$

$= P(-1.2 \leq S \leq -0.8)$

$= F_S(-0.8) - F_S(-1.2) = Q(0.8) - Q(1.2)$

$= 0.2119 - 0.1151 = 0.0968$

ONLY 9.68%.

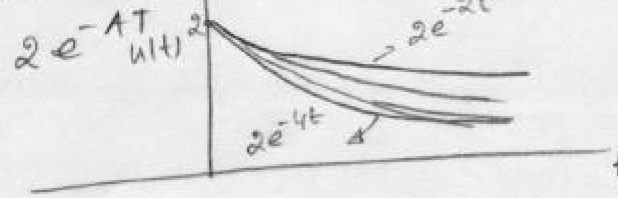
(b) $P(SHELL \geq 3200)$

$= P\left(SHELL \geq \frac{3200-3000}{100}\right) = P(S \geq 2)$

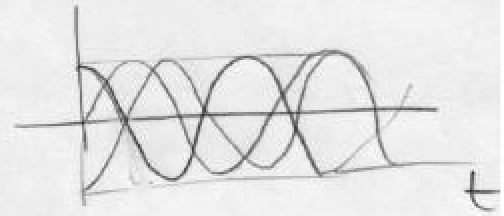
$= 0.0228$

5 $\qquad X(t) = 2e^{-At}\sin(\omega t+\Phi)\,u(t)$ $\qquad$ (6)

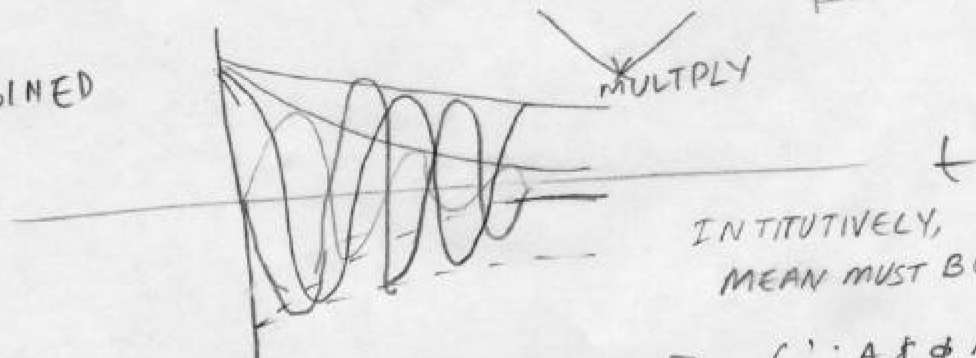
REALIZATIONS OF $\qquad$ $A: U(2,4)$



REALIZATIONS OF $\sin(\omega t+\phi)$



COMBINED



INTITUTIVELY, MEAN MUST BE ZERO

$$E[X(t)] = 2E[e^{-At}]\,E[\sin(\omega t+\Phi)]\,u(t) \quad (\because A \,\&\, \Phi \text{ ARE INDEPENDENT})$$

$$E[e^{-At}] = \int_2^4 \frac{1}{2} e^{-at}\,da = \frac{1}{-2t}\left[e^{-at}\right]_2^4 = \frac{1}{-2t}\left(e^{-4t} - e^{-2t}\right)$$

$$E[\sin(\omega t+\Phi)] = \int_{-\pi}^{\pi} \frac{1}{2\pi}\sin(\omega t+\Phi)\,d\phi = \frac{1}{2\pi}\cos(\omega t+\Phi)\Big|_{-\pi}^{\pi} = 0$$

$E[X(t)] = 0$ CONSTANT

$$\text{AUTO-COVARIANCE} = E\left[(X(t_1) - \mu_{X(t_1)})(X(t_2) - \mu_{X(t_2)})\right]$$

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] \quad (\because \text{MEAN}=0)$$

$$= \int_2^4\int_{-\pi}^{\pi} \left(2e^{-at_1}\sin(\omega t_1+\phi)\right)\left(2e^{-at_2}\sin(\omega t_2+\phi)\right)\left(\frac{1}{4\pi}\right) da\,d\phi$$

$$= \frac{1}{\pi}\left(\int_2^4 e^{-a(t_1+t_2)}\,da\right)\int_{-\pi}^{\pi} \overbrace{\sin(\omega t_1+\Phi)\sin(\omega t_2+\Phi)}^{\frac{1}{2}\left(\cos(\omega t_1-\omega t_2) - \cos(\omega(t_1+t_2)+2\Phi)\right)}\,d\phi$$

$$= \frac{1}{2\pi}\left(\frac{e^{-a(t_1+t_2)}}{t_1+t_2}\Bigg|_2^4\right)\left(\cos\omega(t_1-t_2)(2\pi) + \underbrace{\cancel{\frac{\sin((\omega t_1+\omega t_2)+2\Phi)}{2}\Bigg|_{-\pi}^{\pi}}}_{=0}\right)$$

$$= \frac{\cos\omega(t_1-t_2)}{\pi(t_1+t_2)}\left(e^{-4(t_1+t_2)} - e^{-2(t_1+t_2)}\right)$$

DEPENDS ON $t_1 - t_2$ AND $t_1 + t_2$

$\therefore$ CANNOT BE WSS.